Free Flexural Vibration Response of Integrally-Stiffened and/or Stepped-Thickness Composite Plates or Panels

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(Received 29 September 2012; accepted 21 May 2013)

This study is mainly concerned with a general approach to the theoretical analysis and the solution of the free vibration response of integrally-stiffened and/or stepped-thickness plates or panels with one or more integral plate stiffeners. In general, the Stiffened System is considered to be composed of dissimilar Orthotropic Mindlin Plates with unequal thicknesses. The dynamic governing equations of the individual plate elements of the system and the stress resultant-displacement expressions are combined and algebraically manipulated. These operations lead to the new Governing System of the First Order Ordinary Differential Equations in state vector forms. The new governing system of equations facilitates the direct application of the present method of solution, namely, the Modified Transfer Matrix Method (MTMM) (with Interpolation Polynomials). As shown in the present study, the MTMM is sufficiently general to handle the free vibration response of the stiffened system (with, at least, one or up to three or four Integral Plate Stiffeners). The present analysis and the method of solution are applied to the typical stiffened plate or panel system with two integral plate stiffeners. The mode shapes with their natural frequencies are presented for orthotropic composite cases and for several sets of support conditions. As an additional example, the case of the stiffened plate or panel system with three integral plate stiffeners is also considered and is shown in terms of the mode shapes and their natural frequencies for several sets of the boundary conditions. Also, some parametric studies of the natural frequencies versus the aspect ratio, stiffener thickness ratio, stiffener length (or width) ratio and the bending stiffness ratio are investigated and are graphically presented.

1. INTRODUCTORY REMARKS AND BRIEF REVIEW

The so-called integrally-stiffened and/or stepped-thickness plate or panel systems of various configurations are primarily used in air and space flight vehicle structures and substructures. They may also be utilized in high speed hydrodynamic vehicle structural systems.1-3 Their applications in engineering are due to their advantageous properties of light-dead-weight, freedom from mechanical (riveted, bolted, or welded) connections, and of their favourable stiffness characteristics in appropriate places. A typical and well-known example is their utilization as aircraft wing cover panels with multi-stiffeners.3

These integrally-stiffened plate or panel systems are manufactured by means of the CAD-CAM process as one-piece plate systems out of a solid (or raw stock) of advanced Metal/Alloy plates. In some applications, they may also be manufactured as one-piece advanced composite plate systems with some stepped thicknesses.2,3

The integrally-stiffened plate or panel systems may generally be categorized or grouped in terms of four main groups as shown in Fig. 1. For the purposes of the present study, the Group I systems may further be organized in terms of the number of steps in their configurations, as shown in Fig. 2. In the present study, the integrally-stiffened plate or panel systems (one-step, two-step, three-step, four-step, etc., and all in one direction) are to be analysed by means of a general approach to their free dynamic response.

Some significant research studies on the aforementioned integrally-stiffened and/or stepped-thickness plates are available in world-wide scientific and engineering literature.4-19 In this brief literature survey, one may include the analytic solutions,4,5,10,17,19 The Raleigh-Ritz Method,7,9,11,12 the Finite Element Method (FEM),18 the Finite Strip Method (FSM),13,15 the Kantrovich Method,16 and the Superposition Method.16,18,20,21 have all been studied. More recently, there appeared some studies by Yuceoglu et al.,20-24 which employed the Modified Transfer Matrix Method (MTMM).20-33

The main concern of this study is to present a general approach to the free vibration response of integrally-stiffened and/or stepped-thickness, rectangular Mindlin plates or panels. Thus, the present work aims to achieve two objectives: (1) A general theoretical analysis, and (2) A general method of solution consistent with (1) for the various types of the integrally-stiffened plate or panel systems as defined and shown in Fig. 2.

In the present theoretical analysis, as a general approach, the integrally-stiffened and/or stepped-thickness plate or panel system of Group I, Type 4 (or Four-Step case) in Fig. 3 will be considered. Later on, it will be shown that the Lower-Step or Higher-Step cases may easily be obtained from the aforementioned case. Thus, the Group I, Type 4 (or Four-Step case) is presented in terms of its general configuration, material directions, coordinate systems, and the longitudinal cross-section in Fig. 3. Each stiffened plate system is assumed to be a combination of Mindlin plates34 with dissimilar orthotropic material properties and unequal thicknesses. In both figures, each plate or panel system is considered to be simply supported at \( x = 0, a \), while the boundary conditions in the \( y \)-direction can be arbitrarily specified within the Mindlin plate theory,34 which is one of the First Order Shear Deformation Plate Theories (FSDPTs).34,35
The importance of the transverse shear deformations on the dynamic response of plates are pointed out by Whitney and Pagano,\textsuperscript{36} and also by Librescu et al.\textsuperscript{37} A brief review of some of the various Higher Order Shear Deformation Plate Theories (HSDPTs) can be found in Subramanian.\textsuperscript{38}

### 2. GENERAL ANALYSIS AND DERIVATION OF GOVERNING EQUATIONS

The first step in the present theoretical analysis is to apply the Domain Decomposition technique for the entire plate or panel system under consideration. Thus, depending on a particular type, the plate elements of the system are considered to form Part I, Part II, Part III, etc. regions (or domains) as shown in the longitudinal cross-section of the stiffened system of Fig. 3.

In the second step, the sets of the dynamic equations of the orthotropic Mindlin plates and also the stress resultants-displacement expressions are taken into account as given in Appendix A. The entire system of the governing partial differential equations (PDEs), in a special form (which is very suitable for the present method of solution—i.e., MTMM), is presented in Appendix B. The aforementioned governing PDEs may be written in a compact matrix form, in terms of the state vectors of each part in a type or problem under consideration here.

Hence, in this study, the Group I and Type 4 (or Four-Step Case) is to be considered (see also Fig. 3). Then, referring to Appendix B:

- In Part I region (or Left Plate Stiffener)
  \[
  \frac{1}{l_1} \frac{\partial}{\partial \xi_1} \begin{bmatrix} Y^{(1)} \end{bmatrix} = \left[ C \left( \frac{\partial}{\partial \eta}, \frac{\partial^2}{\partial t^2}, \ldots \right) \right] \begin{bmatrix} Y^{(1)}(\xi_1, \eta) \end{bmatrix};
  \]

- In Part II region (or Right Plate Stiffener)
  \[
  \frac{1}{l_2} \frac{\partial}{\partial \xi_2} \begin{bmatrix} Y^{(2)} \end{bmatrix} = \left[ D \left( \frac{\partial}{\partial \eta}, \frac{\partial^2}{\partial t^2}, \ldots \right) \right] \begin{bmatrix} Y^{(2)}(\xi_2, \eta) \end{bmatrix};
  \]
The next step in the analysis is the Non-Dimensionalization procedure of the governing PDEs of Eqs. (1)–(5) in each of their respective regions. For this purpose the plate element of Part I is chosen as the reference plate. Therefore, \( B_{11}(1) \), \( h_1 \), \( \rho_1 \), and \( a \) are chosen as the main (or reference) quantities (or parameters). Additionally, \( l_i \), \( l_{ii} \), \( h_{ii} \), \( l_{iv} \), \( l_{v} \) are selected as the length reference parameters in the \( y \)-direction in each part (or region), respectively. All other quantities are non-dimensionalized with respect to these parameters.

The dimensionless coordinates in Part I, Part II, ..., Part V regions, respectively, are

\[
\eta = x/a; \quad \xi_1 = y/l_1; \quad (\text{in Part I})
\]
\[
\eta = x/a; \quad \xi_2 = y/l_2; \quad (\text{in Part II})
\]
\[
\eta = x/a; \quad \xi_3 = y/l_3; \quad (\text{in Part III})
\]
\[
\eta = x/a; \quad \xi_4 = y/l_4; \quad (\text{in Part IV})
\]
\[
\eta = x/a; \quad \xi_5 = y/l_5; \quad (\text{in Part V})
\]

(7)

The dimensionless parameters related to the orthotropic elastic constant of plate elements are

\[
\mathcal{B}_{ik}^{(j)} = B_{ik}^{(j)} / B_{11}^{(1)}; \quad (j = 1, 2, 3, \ldots \text{ and } i, k = 1, 2, \ldots);
\]
\[
\mathcal{B}_{ii}^{(j)} = B_{ii}^{(j)} / B_{11}^{(1)}; \quad (j = 1, 2, 3, \ldots \text{ and } l = 1, 2, \ldots);
\]

(8) and the dimensionless parameters related to the densities and the geometry of the plates are

\[
\rho_2 = \frac{\rho_2}{\rho_1}; \quad \rho_3 = \frac{\rho_3}{\rho_1}; \quad \rho_4 = \frac{\rho_4}{\rho_1}; \quad \rho_5 = \frac{\rho_5}{\rho_1}; \quad \rho_1 = \frac{\rho_1}{\rho_1} = 1;
\]
\[
\mathcal{T}_{ii} = \frac{t_{ii}}{a}; \quad \mathcal{T}_{iii} = \frac{t_{iii}}{a}; \quad \mathcal{T}_{iv} = \frac{t_{iv}}{a}; \quad \mathcal{T}_{v} = \frac{t_{v}}{a}; \quad \mathcal{T}_{iv} = \frac{t_{iv}}{a}
\]
\[
\mathcal{h}_2 = \frac{h_2}{h_1}; \quad \mathcal{h}_3 = \frac{h_3}{h_1}; \quad \mathcal{h}_4 = \frac{h_4}{h_1}; \quad \mathcal{h}_5 = \frac{h_5}{h_1}; \quad \mathcal{h}_1 = \frac{h_1}{h_1} = 1;
\]

(9)

where \( a \) is the width of the entire plate system.

The dimensionless natural frequency parameter \( \mathcal{\omega}_{mn} \) of the entire stiffened plate or panel system, (i.e., Type 4 or four-step case)) is

\[
\mathcal{\omega}_{mn} = \frac{\rho_1 a^4 m^2}{h_1^2 B_{11}(1)}; \quad \mathcal{\Omega} = \mathcal{\omega}_{mn}; \quad (m, n = 1, 2, 3, \ldots).
\]

(10)

Here, the non-dimensional natural frequencies \( \mathcal{\omega}_{mn} \) are arranged in the ascending order of \( \mathcal{\Omega}_1 < \mathcal{\Omega}_2 < \mathcal{\Omega}_3 < \ldots \). They may be obtained once \( m \) is assigned and \( n \) is calculated accordingly.

Now, considering the simple support conditions at \( x = 0, a \) in the \( x \)-direction, then the generalized displacements and the stress resultants can be expressed in the Classical Levy’s Method in Fourier series in each part (or region for each plate element). These series are given in Appendix C. Thus, by simply substituting Classical Levy’s Solutions into the system of Eqs. (1)–(5), and making use of some algebraic manipulations and combinations, the governing system of the first order ordinary differential equations in the state-vector forms are obtained. Thus, for the Type 4 or four-step case:

- In Part I region (or Left Plate Stiffener)

\[
\frac{d}{d\xi_1} \{\mathcal{Y}_{mn}^{(1)}\} = \mathcal{B}_{mn}^{(2)} \{\mathcal{Y}_{mn}^{(1)}\}; \quad (0 \leq \xi_1 \leq 1);
\]

(11)

with the continuity conditions at \( \xi_1 = 0, 1 \).
The above dimensionless coefficient matrices replace the following quantities as shown in Fig. 4, may be considered. For this purpose, one may in the theoretical analysis of the problems considered here. It is important to recall that the reduced system of Eqs. (11)–(15) is coupled by means of the continuity conditions at the interfaces of the plate elements of the system. At this point, the Initial Value and Boundary Value Problem of the free dynamic response of the stiffened plate or panel system under study are now reduced or converted to the so-called Two-Point Boundary Value Problem of Mechanics and Physics in terms of the governing system of Eqs. (11)–(15). This is a very important step in the theoretical analysis of the problems considered here.

In order to show that the present analytical formulation is quite general, the Group I and Type 2 (or Two-Step case) of Fig. 4, can be reduced to the Group I and Type 2 (or Two-Step case) of Fig. 4. Hence:

\[
\begin{align*}
\frac{d}{d\xi_I} \left\{ \mathbf{Y}_{mn}^{(1)} \right\} &= \left[ \mathbf{C} \right] \left\{ \mathbf{Y}_{mn}^{(1)} \right\} \quad (0 \leq \xi_I \leq 1); \\
\frac{d}{d\xi_{II}} \left\{ \mathbf{Y}_{mn}^{(2)} \right\} &= \left[ \mathbf{D} \right] \left\{ \mathbf{Y}_{mn}^{(2)} \right\} \quad (0 \leq \xi_{II} \leq 1); \\
\frac{d}{d\xi_{III}} \left\{ \mathbf{Y}_{mn}^{(3)} \right\} &= \left[ \mathbf{E} \right] \left\{ \mathbf{Y}_{mn}^{(3)} \right\} \quad (0 \leq \xi_{III} \leq 1); \\
\frac{d}{d\xi_{IV}} \left\{ \mathbf{Y}_{mn}^{(4)} \right\} &= \left[ \mathbf{F} \right] \left\{ \mathbf{Y}_{mn}^{(4)} \right\} \quad (0 \leq \xi_{IV} \leq 1); \\
\frac{d}{d\xi_{V}} \left\{ \mathbf{Y}_{mn}^{(5)} \right\} &= \left[ \mathbf{G} \right] \left\{ \mathbf{Y}_{mn}^{(5)} \right\} \quad (0 \leq \xi_{V} \leq 1).
\end{align*}
\]

Making use of Eqs. (17) and (18) and inserting these into the governing system of the First Order Ordinary Differential Equations (ODEs) in Eqs. (11)–(15), they can be reduced to the Group I and Type 2 (or Two-Step case) of Fig. 4. Hence:

\[
\begin{align*}
\xi_I &\rightarrow \xi_I; \quad \eta = x/a; \\
\xi_{II} &\rightarrow \xi_{II}; \quad \eta = x/a; \\
\xi_{III} &\rightarrow \xi_{III}; \quad \eta = x/a; \\
\xi_{IV} &\rightarrow \xi_{IV}; \quad \eta = x/a; \\
\xi_{V} &\rightarrow 0; \quad \eta = x/a.
\end{align*}
\]

\[
\begin{align*}
\mathbf{C} &\rightarrow \mathbf{C}; \\
\mathbf{D} &\rightarrow \mathbf{D}; \\
\mathbf{E} &\rightarrow \mathbf{E}; \\
\mathbf{G} &\rightarrow 0; \quad (17)
\end{align*}
\]
In the above, the dimensionless fundamental dependent variables or the state vectors are now expressed as

\[ \left\{ \psi_{mn}^{(j)}(\xi_k) \right\} = \left\{ \psi_{mn,1}^{(j)}, \psi_{mn,2}^{(j)}, m_{mn,1}^{(j)}, m_{mn,2}^{(j)} \right\}^T ; \]

\( j = 1, 2, 3; \quad k = I, II, III \). \hspace{1cm} (22)

Again, it should be noted that the Initial Value and Boundary Value Problem corresponding to the configuration in Fig. 4 is finally reduced to a Two-Point Boundary Value Problem of Mechanics and Physics in terms of the new set of the governing systems of the First Order ODEs in Eqs. (19)–(21) for the Group I and Type 2 (or Two-Step case); see also Yuceoglu et al. \[25,33\]

In a similar manner, one can write the governing system of the First Order ODEs for the Group I Type 6 (Six-Step case) (or higher order cases).

3. GENERAL METHOD OF SOLUTION AND NUMERICAL PROCEDURE

The solution technique to be employed is the Modified Transfer Matrix Method (MTMM) (with Interpolation Polynomials). The state vector forms of the governing system of equations as given in Eqs. (11)–(15), facilitates the present method of solution. This semi-analytical and numerical solution procedure is a combination of the Classical Levy’s Method, the Transfer Matrix Method, and the Integrating Matrix Method (with Interpolation Polynomials).

The aforementioned method has been developed and been efficiently utilized for some classes of plate and shallow shell vibration problems by Yuceoglu et al. \[25-33,39,40\] It can be found in more detail in Yuceoglu and Özciriyes, \[25-33\] Yuceoglu et al. \[39\], and in earlier versions in Yuceoglu et al. \[40\].

The two other more recently developed versions of this method are the MTMM (with Chebyshev Polynomials) and the MTMM (with Eigenvalue Approach). These recent versions by Yuceoglu and Özciriyes \[39\] are also highly accurate and more suitable for computing the higher natural frequencies and modes (higher than the sixth mode and up to fifteen or higher modes).

In the present study, in connection with the stiffened plates or panels, the MTMM (with Interpolation Polynomials) is to be used. Referring to the previous studies on the integrally-stiffened and/or stepped-thickness plates by Yuceoglu et al., \[20-24\], the main steps of the method are briefly explained next. The very first step is to discretize the state vectors and the Coefficient Matrices as defined in Eqs. (11)–(15). The discretization procedure is simply to divide Part I, Part II, ..., Part V regions, into sufficient number of points (or stations) along \( \xi_1, \xi_2, ..., \xi_V \) directions, respectively. Following the discretization, in the second step, the discretized versions of the governing system of Eqs. (11)–(15), are pre-multiplied by the appropriate Global Integrating Matrices \[ L_1, L_2, ..., L_V \] in their respective parts (or regions). The global integrating matrices include integrating sub-matrices \( L \) for each part, respectively. Then, in the Part I region (Left Plate Stiffener), the following equation is obtained

\[ \left\{ \dot{\psi}_{mn}^{(1)} \right\} - \left\{ \dot{\psi}_{mn}^{(1)} \right\}_{\xi_1=0} = \left[ L_1 \right] \left[ \dot{C} \right] \left\{ \dot{\psi}_{mn}^{(1)} \right\} . \] \hspace{1cm} (23)

In Eq. (23), the discretizations are shown by the \( \cdot \) sign on matrices along the \( \xi \)-directions; the subscript \( \xi_1 = 0 \) means that the matrix is evaluated at the initial end point \( \xi_1 = 0 \). The above expression can be further rearranged between the initial end point \( \xi_1 = 0 \) and a general station \( \xi_2 > 0 \) in the Part I region. Then, dropping the \( mn \) subscripts for convenience, one obtains:

- **In Part I region (Left Plate Stiffener)**
  \[ \left\{ \dot{\psi}_{\xi_1}^{(1)} \right\} = \left[ U \right] \left\{ \dot{\psi}_{\xi_1=0}^{(1)} \right\} ; \]
  \[ U = \left[ \left[ I - [L_1] \left[ C \right] \right]^{-1} \right] ; \] \hspace{1cm} (24)

where \( \left[ U \right] \) is the discretized Global Modified Transfer Matrix for Part I region and \( [I] \) is the unit matrix. Similarly, one can write the following expressions for the Part II, Part III, ..., Part V regions:

- **In Part II region (Right Plate Stiffener)**
  \[ \left\{ \dot{\psi}_{\xi_2}^{(2)} \right\} = \left[ V \right] \left\{ \dot{\psi}_{\xi_2=0}^{(2)} \right\} ; \]
  \[ V = \left[ \left[ I - [L_2] \left[ D \right] \right]^{-1} \right] ; \] \hspace{1cm} (25)

- **In Part III region (Far Left Plate)**
  \[ \left\{ \dot{\psi}_{\xi_3}^{(3)} \right\} = \left[ W \right] \left\{ \dot{\psi}_{\xi_3=0}^{(3)} \right\} ; \]
  \[ W = \left[ \left[ I - [L_3] \left[ E \right] \right]^{-1} \right] ; \] \hspace{1cm} (26)

- **In Part IV region (Right Plate)**
  \[ \left\{ \dot{\psi}_{\xi_4}^{(4)} \right\} = \left[ S \right] \left\{ \dot{\psi}_{\xi_4=0}^{(4)} \right\} ; \]
  \[ S = \left[ \left[ I - [L_4] \left[ F \right] \right]^{-1} \right] ; \] \hspace{1cm} (27)

- **In Part V region (Middle Plate)**
  \[ \left\{ \dot{\psi}_{\xi_5}^{(5)} \right\} = \left[ T \right] \left\{ \dot{\psi}_{\xi_5=0}^{(5)} \right\} ; \]
  \[ T = \left[ \left[ I - [L_5] \left[ G \right] \right]^{-1} \right] ; \] \hspace{1cm} (28)

where \( \left[ U \right], \left[ V \right], \left[ W \right], \left[ S \right], \) and \( \left[ T \right] \) are the discretized forms of the Global Modified Transfer Matrices for their respective parts (or regions), the matrices \( [L_1], [L_2], [L_3], [L_4], \) and \( [L_5] \) are the Integrating Matrices with roman subscripts indicating the particular part (or region) in which they operate. As mentioned before, they include the Integrating Sub-Matrices \( [L] \).

Here, one can express the relations between the state vectors at the initial end points \( \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 = 0 \), and the state vectors at the final end points \( \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 = 1 \) in each Part I, Part II, ..., Part V regions, respectively. Thus, dropping the \( \cdot \) sign for convenience, the following equation systems are obtained:
Some further combinations and manipulations yield the final end point. They read from left to right along the system: 

\[ \{ \mathbf{Y}_{\xi_1}^{(1)} \} = \{ \mathbf{U}_{1,1} \}_{01} \{ \mathbf{Y}_{\xi_0}^{(1)} \} ; \quad (0 \leq \xi_1 \leq 1) \]  \hspace{1cm} (29)

In Part II region (Right Plate Stiffener)

\[ \{ \mathbf{Y}_{\xi_2}^{(2)} \} = \{ \mathbf{V}_{1,1} \}_{01} \{ \mathbf{Y}_{\xi_0}^{(2)} \} ; \quad (0 \leq \xi_2 \leq 1) \]  \hspace{1cm} (30)

In Part III region (Far Left Plate Element)

\[ \{ \mathbf{Y}_{\xi_3}^{(3)} \} = \{ \mathbf{W}_{1,1} \}_{01} \{ \mathbf{Y}_{\xi_0}^{(3)} \} ; \quad (0 \leq \xi_3 \leq 1) \]  \hspace{1cm} (31)

In Part IV region (Far Right Plate Element)

\[ \{ \mathbf{Y}_{\xi_4}^{(4)} \} = \{ \mathbf{S}_{1,1} \}_{01} \{ \mathbf{Y}_{\xi_0}^{(4)} \} ; \quad (0 \leq \xi_4 \leq 1) \]  \hspace{1cm} (32)

In Part V region (Middle Plate Element)

\[ \{ \mathbf{Y}_{\xi_5}^{(5)} \} = \{ \mathbf{T}_{1,1} \}_{01} \{ \mathbf{Y}_{\xi_0}^{(5)} \} ; \quad (0 \leq \xi_5 \leq 1) \]  \hspace{1cm} (33)

where the subscripts 0 mean that the final forms of the discretized Global Modified Transfer Matrices \( \{ \mathbf{U} \}, \{ \mathbf{V} \}, \{ \mathbf{W} \}, \{ \mathbf{S} \}, \) and \( \{ \mathbf{T} \) are transferring the state vectors from the initial end point 0 to the final end point 1 in each part (or region), respectively.

At this point, the continuity conditions in terms of the state vectors between (Part III, Part IV), (Part I, Part V), (Part I, Part II), etc. are inserted into the above Eqs. (29) through (33). Some further combinations and manipulations yield the final equation for the entire panel system as

\[ \{ \mathbf{Y}_{\xi_1}^{(1)} \} = \{ \mathbf{S}_{1,1} \}_{01} \{ \mathbf{Y}_{\xi_0}^{(1)} \} ; \quad (0 \leq \xi_1 \leq 1) \]  \hspace{1cm} (34)

It is important to observe here that \( \{ \mathbf{Q} \}_{01} \) is the final form of the discretized Overall Global Modified Transfer Matrix. This matrix transfers the above state variables from the initial end point (far left end support) \( \xi_0 = 0 \) to the final end point (far right end support) \( \xi_0 = 1 \) of the entire plate or panel system of Group I and Type 4 (or Four-Step case). The above Overall Global Modified Transfer Matrix \( \{ \mathbf{Q} \}_{01} \) can further be reduced to a \((3 \times 3)\) matrix by making use of or inserting of the Boundary Conditions at the initial end point (or far left end support) \( \xi_0 = 0 \) and the final right end point (far right end support) \( \xi_0 = 1 \). This operation yields the following:

\[ \{ \mathbf{Q}_0 (\varpi_{mn}) \} \{ \mathbf{Y}_0 \} = \{ 0 \} ; \quad \text{Determinant of Coeff. Matrix } \{ \mathbf{C}_0 (\varpi_{mn}) \} = 0; \]  \hspace{1cm} (35)

where the above Coefficient Matrix \( \{ \mathbf{C}_0 \} \) implicitly includes the unknown natural frequency parameter \( \varpi_{mn} \). Thus, \( \{ \mathbf{Y}_0 \} \) is not necessarily zero, then the determinant of the Coefficient Matrix must be equal to zero, yielding the polynomial whose roots are the natural frequencies of the entire plate or panel system:

\[ \tilde{\Omega} = \varpi_{mn}; \quad (m, n = 1, 2, 3, \ldots) \]

\[ \tilde{\Omega}_1 \times \tilde{\Omega}_2 \times \tilde{\Omega}_3 \times \ldots ; \]  \hspace{1cm} (36)

where the dimensionless natural frequencies \( \varpi_{mn} \) are computed by searching the roots numerically on the basis of the given \( m \) and the assigned \( n \) values. After then, they are sequenced or reorganized (or renamed) according to their magnitudes as shown above with a single subscript in Eq. (36).

4. SOME NUMERICAL RESULTS AND CONCLUSIONS

The present general approach to the analytical formulation and the method of solution are applied to the problem of the free dynamic response of integrally-stiffened and/or stepped-thickness rectangular plate or panel system with two plate stiffeners given in Fig. 3; i.e., Group I and Type 4 (or Four-Step case). The present analytical formulation can easily be extended to the Type 6 (or the Six-Step case).

In the set of the present numerical examples, the entire plate or panel system is assumed to be made of orthotropic composite system (Orthotropic case). The two and three plate stiffeners are considered as made of Graphite-Epoxy and the plate elements of the system are chosen as Kevlar-Epoxy. Thus, it is possible to compare, in terms of the mode shapes and the corresponding natural frequencies, and consequently reach some important conclusions.

The material and the dimensional characteristics of orthotropic composite cases are given in Table 1. The boundary conditions on all figures are shown only as the far left and the far right support conditions of the whole stiffened system. They read from left to right along the \( y \)-direction as \( C = \) Clamped, \( S = \) Simple, and \( F = \) Free support conditions.
In Figs. 5–8 for the Orthotropic Composite Plate or Panel System, the mode shapes and their natural frequencies are shown for the (CC) and (CS) boundary conditions, respectively. The symmetric and/or skew-symmetric mode shapes are observed when the Complete Symmetry conditions hold, as in Fig. 5. Otherwise, in Fig. 6, this is not the case.

The significant effects of several important parameters on the dimensionless natural frequencies of the present stiffened system are also investigated. These parameters are the aspect ratio $a/L$, the stiffeners length (or width) ratio $h_1(=l_1)/L$, the stiffener thickness ratio $h_3(=h_4=h_5)/h_2(=h_2)$ and also the bending stiffness ratio $D_{22}^{(3)}/D_{22}^{(1)}(=D_{22}^{(2)})$. Each individual parameter versus the dimensionless natural frequencies are computed and presented in the next set of Figs. 9–12.

The aspect ratio $a/L$ is the parameter which significantly affects the natural frequencies of the entire system. Thus, in Fig. 9, the natural frequency curves are highly non-linear and they increase sharply as the aspect ratio $a/L$ increases, regardless of the support conditions of the system.

The stiffeners length (or width) ratio $h_1(=l_1)/L$ versus the dimensionless natural frequencies for each set of the support conditions are shown in Fig. 10. Regardless of the support conditions, the main characteristic observed is that the dimensionless natural frequencies exhibit linear behavior and they increase gradually with the increasing parameter under study.

The bending stiffness ratio $D_{22}^{(3)}/D_{22}^{(1)}(=D_{22}^{(2)})$ versus the dimensionless natural frequencies are shown in Fig. 12. Regardless of the boundary conditions the natural frequencies increase gradually and almost linearly.

On the basis of the present Governing Equations and the present Solution Method, it can further be concluded that these are fairly general for the Integrally-Stiffened and/or Stepped-Thickness Plates Systems considered in this study.

REFERENCES


Figure 6. Mode shapes and dimensionless natural frequencies of integrally-stiffened and/or stepped-thickness rectangular plate or panel system with two plate stiffeners (orthotropic case). Plate 1 = Plate 2 = Graphite-Epoxy, Plate 3 = Plate 4 = Plate 5 = Kevlar-Epoxy; l1 = 0.15 m, l2 = 0.15 m, l3 = 0.233 m, h1 = 0.233 m, h2 = 0.234 m; a = 0.50 m, h3 = h4 = h5 = 0.02 m, L = 1.00 m; boundary conditions in y-direction CS.


Figure 7. Mode shapes and dimensionless natural frequencies of integrally-stiffened and/or stepped-thickness plate or panel system with three plate stiffeners (orthotropic case). Plate 1 = Plate 2 = Plate 3 = Graphite-Epoxy, Plate 4 = Plate 5 = Plate 6 = Plate 7 = Kevlar-Epoxy; \( l_1 = l_2 = l_3 = 0.140 \text{ m}, \ l_4 = l_5 = l_6 = l_7 = 0.145 \text{ m}; \ a = 0.50 \text{ m}, \ h_1 = h_2 = h_3 = 0.04 \text{ m}, \ h_4 = h_5 = h_6 = h_7 = 0.02 \text{ m}, \ L = 1.00 \text{ m}; \) boundary conditions in \( y \)-direction CC.


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Figure 8. Mode shapes and dimensionless natural frequencies of integrally-stiffened and/or stepped-thickness plate or panel system with three plate stiffeners (orthotropic case). Plate 1 = Plate 2 = Plate 3 = Graphite-Epoxy, Plate 4 = Plate 5 = Plate 6 = Plate 7 = Kevlar-Epoxy; \( l_1 = l_II = l_III = 0.140 \text{ m} \); \( l_IV = l_V = l_VI = l_VII = 0.145 \text{ m} \); \( a = 0.50 \text{ m} \); \( h_1 = h_2 = h_3 = 0.04 \text{ m} \); \( h_4 = h_5 = h_6 = h_7 = 0.02 \text{ m} \); \( L = 1.00 \text{ m} \); boundary conditions in \( y \)-direction CS.


Figure 9. Dimensionless natural frequencies $\Omega$ versus aspect ratio $a/L$ in integrally-stiffened and/or stepped thickness plate or panel system with two stiffeners (orthotropic case). Plate 1 = Plate 2 = Graphite-Epoxy, Plate 3 = Plate 4 = Plate 5 = Kevlar-Epoxy; $h_1 = 0.2$ m, $h_2 = 0.2$ m, $t_{in} = 0.2$ m, $h_V = 0.2$ m, $h_T = 0.2$ m; $a$ varies, $h_1 = h_2 = 0.04$ m, $h_3 = h_4 = h_5 = 0.02$ m, $L = 1.00$ m; (a) boundary conditions in $y$-directions CC, (b) boundary conditions in $y$-directions CS.


Figure 11. Dimensionless natural frequencies $\Omega$ versus thickness ratio $h_3(=h_4 = h_5) / h_1(= h_2 = 0.04)$ in integrally-stiffened and/or stepped thickness plate or panel system with two stiffeners (orthotropic case). Plate 1 = Plate 2 = Graphite-Epoxy, Plate 3 = Plate 4 = Plate 5 = Kevlar-Epoxy; $l_1 = 0.2\ m$, $l_{II} = 0.2\ m$, $l_{III} = 0.2\ m$, $l_{IV} = 0.2\ m$, $l_{V} = 0.2\ m$; $a = 0.50\ m$, $h_1 = h_2 = 0.04\ m$, $h_3 = h_4 = h_5$ varies, $L = 1.00\ m$; (a) boundary conditions in $y$-directions CC, (b) boundary conditions in $y$-directions CS.

APPENDIX A: MINDLIN PLATE THEORY AS APPLIED TO ORTHOTROPIC PLATES

Equations of Motion of Mindlin Plates

\[
\begin{align*}
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xz}}{\partial y} - Q_x &+ \frac{h}{2}(q^+_x + q^-_x) = \frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2}; \\
\frac{\partial M_{xz}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &+ \frac{h}{2}(q^+_y + q^-_y) = \frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2}; \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + (q^+_x + q^-_x) = \rho h \frac{\partial^2 w}{\partial x^2}; \\
\end{align*}
\]

(A.1)

where $q$-s are upper and lower loads or surface stresses, respectively.

Stress Resultant-Displacement Relations (in terms of Elastic Constants)

\[
\begin{align*}
M_x &= \frac{h^3}{12} \left( B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} \right); \\
Q_x &= \kappa^2 h B_{55} \left( \psi_x + \frac{\partial w}{\partial x} \right); \\
M_y &= \frac{h^3}{12} \left( B_{21} \frac{\partial \psi_x}{\partial x} + B_{22} \frac{\partial \psi_y}{\partial y} \right); \\
\end{align*}
\]

Stress Resultant-Displacement Relations (in terms of Stiffnesses)

\[
\begin{align*}
M_x &= D_{11} \frac{\partial \psi_x}{\partial x} + D_{12} \frac{\partial \psi_y}{\partial y}; \\
Q_x &= A_{55} \left( \psi_x + \frac{\partial w}{\partial x} \right); \\
M_y &= D_{21} \frac{\partial \psi_x}{\partial x} + D_{22} \frac{\partial \psi_y}{\partial y}; \\
\end{align*}
\]

where $\kappa^2$ terms are the Shear Correction Factors of the Mindlin Plate Theory, and $B$-s are material coefficients in the orthotropic stress-strain relations (Hooke’s Law) such that

\[
\begin{align*}
B_{11} &= \frac{E_x}{1 - \nu_{xy} \nu_{xz}}; & B_{44} &= G_{yz}; \\
B_{22} &= \frac{E_y}{1 - \nu_{yx} \nu_{yz}}; & B_{55} &= G_{xz}; \\
B_{12} &= B_{21} = \nu_{yz} B_{11} = \nu_{xy} B_{22}; & B_{66} &= G_{xy}. \\
\end{align*}
\]

(A.3)
where the bending stiffness $D$-s and the shear stiffness $A$-s are

$$D_{ik} = \frac{h^3 B_{ik}}{12}; \quad (i, k = 1, 2);$$

$$D_{66} = \frac{h^3 B_{66}}{12};$$

$$A_{14} = \kappa^2_B h B_{14}; \quad A_{55} = \kappa^2_B h B_{55}. \quad (A.5)$$

**Mindlin Boundary Conditions**

F (free) : \quad $M_{nt} = M_n = Q_n = 0$;

S (simply supported) : \quad $w = \psi_1 = M_n = 0$;

F (clamped) : \quad $w = \psi_n = \psi_t = 0$; \quad (A.6)

where $n$ and $t$ are normal and tangential directions.

**APPENDIX B: SPECIAL FORM OF GOVERNING PDES FOR PARTS I, II, III, IV, V (ORTHOTROPIC MINDLIN PLATE THEORY)**

$$1 \frac{\partial \psi_y^{(j)}}{\partial \xi} = \frac{1}{B(y_2)} \left( 12 M_y^{(j)} - B(y_2) \frac{1}{a} \frac{\partial \psi_y^{(j)}}{\partial \eta} \right);$$

$$1 \frac{\partial \psi_x^{(j)}}{\partial \eta} = \frac{12 h_j^3 B_{66}^{(j)}}{B(y_2)} M_{xx}^{(j)} - \frac{1}{a} \frac{\partial \psi_y^{(j)}}{\partial \eta};$$

$$1 \frac{\partial M_y^{(j)}}{\partial \xi} = \frac{1}{B(y_2)^2} \frac{\partial}{\partial \eta} \left( \frac{\partial Q_y^{(j)}}{\partial \xi} - \psi_y^{(j)} \right);$$

$$1 \frac{\partial M_x^{(j)}}{\partial \eta} = \frac{\partial}{\partial \eta} \left( \frac{\partial Q_x^{(j)}}{\partial \eta} - \psi_x^{(j)} \right);$$

$$1 \frac{\partial Q_y^{(j)}}{\partial \xi} = \frac{\partial}{\partial \eta} \left( \frac{\partial Q_x^{(j)}}{\partial \xi} - \psi_x^{(j)} \right);$$

where

$$j = 1; \quad k = I; \quad \text{(in Part I)}$$

$$j = 2; \quad k = II; \quad \text{(in Part II)}$$

$$j = 3; \quad k = III; \quad \text{(in Part III)}$$

$$j = 4; \quad k = IV; \quad \text{(in Part IV)}$$

$$j = 5; \quad k = V; \quad \text{(in Part V)} \quad (B.2)$$

and where $q$-s are the surface loads (or stresses) which are identically zero in this case. In the above system of equations, the state vectors of the problem are given as column vectors for Parts I, II, III, . . . V, respectively

$$\left\{ Y^{(j)} (\xi, \eta) \right\} = \left\{ \psi_x^{(j)}, \psi_y^{(j)}, W^{(j)}, M_{y^{(j)}}, M_{y^{(j)}}, Q_{y^{(j)}} \right\}^T; \quad (j = 1, 2, 3, 4, 5; \quad k = I, II, III, IV, V). \quad (B.3)$$

**APPENDIX C: CLASSICAL LEVY’S SOLUTIONS**

Following the Domain Decompositions Technique, the Classical Levy’s Solutions in the $x$-direction, corresponding, to each part or region are given below.

**Displacement and Angles of Rotation**

$$w^{(j)} (\eta, \xi, k, t) = h_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}^{(j)} (\xi_k) \sin(m \pi \eta) e^{i \omega m \eta};$$

$$\psi_x^{(j)} (\eta, \xi, k, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^{(j)} (\xi_k) \cos(m \pi \eta) e^{i \omega m \eta};$$

$$\psi_y^{(j)} (\eta, \xi, k, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^{(j)} (\xi_k) \sin(m \pi \eta) e^{i \omega m \eta}; \quad (C.1)$$

where $j = 1$ for Part I, $j = 2$ for Part II, $j = 3$ for Part III, $j = 4$ for Part IV, $j = 5$ for Part V, $k = I$ for Part I, $k = II$ for Part II, $k = III$ for Part III, $k = IV$ for Part IV, $k = V$ for Part V.

**Stress Resultants**

$$M_x^{(j)} (\eta, \xi, k, t) = h_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}^{(j)} (\xi_k) \sin(m \pi \eta) e^{i \omega m \eta};$$

$$M_y^{(j)} (\eta, \xi, k, t) = h_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}^{(j)} (\xi_k) \cos(m \pi \eta) e^{i \omega m \eta};$$

$$Q_y^{(j)} (\eta, \xi, k, t) = h_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn}^{(j)} (\xi_k) \sin(m \pi \eta) e^{i \omega m \eta};$$

$$Q_x^{(j)} (\eta, \xi, k, t) = h_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn}^{(j)} (\xi_k) \cos(m \pi \eta) e^{i \omega m \eta}; \quad (C.2)$$