A Two-Step Model Reduction Method for a Strap-on Launch Vehicle

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In this paper, a two-step model reduction method is proposed using a strap-on launch vehicle as research object. In this method, the double-compatible free-interface modal synthesis method is first used for the modeling of the system, and the first step of reduction work is done by preserving the lower-order modes of flexible components of the system in the modeling process for the system. Then the second step of reduction is done for the established dynamic model of the system using the modal cost analysis method. After the two steps of order reduction, a low-order dynamic model of the system can be obtained. Simulation results indicate that this low-order model can reflect the characteristics of the original system effectively and the order is low enough as well.

1. INTRODUCTION

It is well known that flexible components are used increasingly in aerospace structures due to limits on the weight of the launch vehicle. Moreover, to deliver more payloads to outer space, there is a trend to fit launch vehicles with long thrusters in order to carry more fuel; this results in the obvious flexibility of the launch vehicle. The introduction of flexibility makes dynamic modelling and the control of the system much more difficult. To describe the dynamic behaviour of the system exactly, flexible characteristics in the system should be adequately taken into account during the modelling process. This results in a high order of the established dynamic model. However, control design and implementation require that the order of the system should be as low as possible. Therefore, model reduction work should be done in order to obtain a low-order model convenient for control design. This low-order model should be able to reflect the dynamic characteristics of the original system, and its order should also be low enough. On the other hand, from the point of dynamic simulation, model order should not be high so as to improve computational efficiency.

For linear structure, lower-order modes play a main role in the dynamic response of the structure. So the lower-order modes are often preserved to form a reduction model to represent the original system approximately. This is the so-called modal truncation technique which has been widely applied in both theoretical research and engineering application. Essentially, the modal synthesis technique is a method used to build a low-order dynamic model for a complex system using the modal truncation technique. In using the modal synthesis method, the complex system is divided into several components reasonably, and the high-order dynamic equation of each component is first established using the finite element method (FEM). Then, the modal truncation technique is applied to the FEM equation in order to obtain a low-order modal dynamic equation of the component. Finally, the dynamic equation of the system is established by doing the modal synthesis for all the components based on the displacement and force compatibility conditions at the interfaces of the components. The first step of the modal reduction of the two step-method proposed in this paper is the local reduction of the system. This is when the order reduction in the modal synthesis method is done for the components in the modelling process for the system. Up to now, some different types of modal synthesis methods have been proposed by researchers according to the different selections of the modal functions of flexible components. The typical ones are the free-interface modal synthesis method and the fixed-interface modal synthesis method. Wang et al. modified the free-interface modal synthesis method and proposed the double-compatible free-interface modal synthesis method, a method that can achieve higher computation precision than the free-interface and fixed-interface modal synthesis methods. The modal synthesis method is applicable to the dynamic modelling of large and complex engineering structures, and its validity has been justified in many practical applications. For example, a space station is mainly composed of several cabins that are sent into space at different times. Generally, a ground test needs to be done for every cabin before sending it into space. However, the overall ground test of a space station is usually impossible. In this case, the modal synthesis method is available to get the characteristics of the space station by synthesizing the data of every cabin.

Other than the modal synthesis method mentioned above, there is another kind of model reduction technique in which the reduction is done after the system model is established. We call this technique the integral reduction of the system, which is the second step of the two-step model reduction method discussed in this paper. The integral reduction technique can be mainly divided into two classes: the dynamic condensation technique and the criteria reduction technique. These two techniques have both been studied deeply by researchers and widely applied in practical engineering. Furthermore, the dynamic condensation technique has been implemented in some commercial dynamics analysis software such as Nastran, Ansys, etc. For a spacecraft, the criteria reduction method is often used. The commonly used criteria include the inertia completeness criteria, the modal cost analysis criteria, and the balanced reduction criteria. Among these criteria, the modal cost analysis criteria has been extensively studied since it is
related to the vibration mode that is most widely used in structural dynamics.\textsuperscript{20–25}

In this paper, a two-step model reduction method is proposed for a strap-on launch vehicle, where the first step of reduction (local reduction) is done using the double-compatible, free-interface modal synthesis method, and the second step (integral reduction) is done by using the modal cost analysis method. By the two steps of reduction, a low-order model of the system can be obtained. The validity of the proposed method is verified numerically at the end of this paper.

2. LOCAL REDUCTION USING THE DOUBLE-COMPATIBLE FREE-INTERFACE MODAL SYNTHESIS METHOD

For a strap-on launch vehicle, the system may be described using a beam model when the system is symmetrical and the vibration of the system is linear. As shown in Fig. 1a, the launch vehicle is composed of a main rocket and four boosters. The four boosters are identical, and they are symmetrically arranged with respect to the main rocket. In this paper, the plane problem of the strap-on launch vehicle is considered as shown in Fig. 1b—where the main rocket is represented by the long beam, and the two boosters are represented by the short beams. The mode synthesis method will be used to establish the dynamic equation of the system. In using the mode synthesis method, the system is divided into three substructures, as shown in Fig. 2. Substructure 1 and Substructure 3 are both connected to the long beam by the two nodes, shown as the big black dots in Fig. 2. It is expected that force is only force acting on the joint of the main rocket and the booster where a bending moment may cause deformation damage to the shell of the main rocket. So, the joints of the long beam and the short beam in Fig. 2 can be regarded as the hinge joints. In other words, the interface forces on the big black node in Fig. 2 only contain a force component and do not contain a bending moment. The three substructures in Fig. 2 will be analysed using the finite element method (FEM) before the modal synthesis is done. The black dot in Fig. 2 represents the FEM node. Each dot has three degrees of freedom (DOFs), namely the transverse and axial displacements and the angle of the cross-section. The parameters $F_1$, $F_2$, and $F_3$ are the thrusts produced by the three motors of the main rocket and the two boosters. They act at the lowest nodes of the three beams, as shown in Fig. 2, and the angle between the thrusts and the axial direction are all $\delta$.

Substructure 1 is considered first. In analysing Substructure 1 using the FEM, it is divided into five sections: $J_1$, $J_{11}$, $J_{12}$, $J_{2}$, and $J_{3}$, as shown in Fig. 2. The vectors of the FEM node coordinates of these five sections are represented by $u_{1j1}$, $u_{1j1}$, $u_{1j1}$, $u_{1j1}$, and $u_{1j1}$, respectively, where $u_{1j1}$, $u_{1j1}$, and $u_{1j1}$ are the internal node coordinates, and $u_{1j1}$ and $u_{1j1}$ are the interface node coordinates. The first number “1” in subscript represents Substructure 1. For example, $u_{1j1}$ means the node-coordinate vector of the $J_2$ section of Substructure 1. This kind of subscript expression will be used throughout this paper.

The FEM dynamic equation of Substructure 1 can be written as

$$M_1 \ddot{u}_1 + K_1 u_1 = B_{1j} f_1 + B_{1j} F_1; \quad (1)$$

where $u_1$ is the node-coordinate vector of Substructure 1, $u_1 = \begin{bmatrix} u_{1j1}^T & u_{1j1}^T & u_{1j1}^T \end{bmatrix}^T$, and $B_{1j}$ is the interface-force vector of Substructure 1 and $B_{1j}$ is the interface-force vector of the two nodes of Substructure 2 connected to Substructure 1, respectively. Since force action only exists on two of the nodes, $f_{1j1}$ and $f_{1j2}$, both only contain the two force components in the transverse and the axial directions, respectively. The parameters $B_{1j}$ and $B_{1j}$ are the Boole indicated matrix of $u_{1j1}$ and $B_{1j}$ is the external force of Substructure 1, namely the thrust of Substructure 1; $B_{1j}$ is the Boole indicated matrix of $f_1$. Since $F_1$ acts at the lowest node of the $J_3$ section, and the angle of inclination is $\delta$, the two elements of $B_{1j}$ corresponding to $F_1$ are $\sin \delta$ and $\cos \delta$, with other elements of $B_{1j}$ being zero.

Rearranging $u_1$ as

$$u_1^T = \begin{bmatrix} u_{1j1}^T & u_{1j1}^T & u_{1j1}^T \end{bmatrix} = \begin{bmatrix} f_{1j1}^T \end{bmatrix}; \quad (2)$$

where $u_{1j1}$ and $u_{1j1}$ are the interface-force vectors of the two nodes of Substructure 1 connected to Substructure 2, respectively. Since force action only exists on two of the nodes, $f_{1j1}$ and $f_{1j2}$, both only contain the two force components in the transverse and the axial directions, respectively. The parameter $B_{1j}$ is the Boole indicated matrix of $f_{1j1}$ and $B_{1j}$ is the external force of Substructure 1, namely the thrust of Substructure 1; $B_{1j}$ is the Boole indicated matrix of $f_1$. Since $F_1$ acts at the lowest node of the $J_3$ section, and the angle of inclination is $\delta$, the two elements of $B_{1j}$ corresponding to $F_1$ are $\sin \delta$ and $\cos \delta$, with other elements of $B_{1j}$ being zero.

$u_1^T = T_1 u_1, T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \quad (3)$
Making the transformation for Eq. (1), we have
\[ M_1^* \ddot{u}_1^* + K_1^* u_1^* = B_{1j}^* f_{1j} + B_{1F}^* F_1; \]
where \( M_1^* = T_1 M_1 T_1^{-1} \), \( K_1^* = T_1 K_1 T_1^{-1} \), \( B_{1j}^* = T_1 B_{1j} \), and \( B_{1F}^* = T_1 B_{1F} \).

Making modal analysis for Substructure 1, we have
\[ u_1^* = \Phi_{1n} p_1 = [\Phi_{1k} \Phi_{1d}] \begin{bmatrix} \Phi_{1k} \end{bmatrix} p_{1k} + \Phi_{1d} p_{1d}; \]
where \( \Phi_{1n} = [\Phi_{1k} \Phi_{1d}] \) is the modal transformation matrix, and \( \Phi_{1k} \) and \( \Phi_{1d} \) are the lower and higher modes of Substructure 1, respectively; \( p_1 = [p_{1k}^T \ p_{1d}^T]^T \) is the modal coordinate vector of Substructure 1, and \( p_{1k} \) and \( p_{1d} \) are the vectors corresponding to \( \Phi_{1k} \) and \( \Phi_{1d} \), respectively. Substituting Eq. (5) into Eq. (4) and left multiplying \( \Phi_{1n}^T \), we have
\[ \ddot{M}_1^* \ddot{p}_1 + \ddot{K}_1^* p_1 = \ddot{B}_{1j}^* f_{1j} + \ddot{B}_{1F}^* F_1; \]
where
\[ \ddot{M}_1^* = \begin{bmatrix} \ddot{M}_{1k} & 0 \\ \Phi_{1k}^T \ddot{M}_{1d} \end{bmatrix} \]
and
\[ \ddot{K}_1^* = \begin{bmatrix} \ddot{K}_{1k} & 0 \\ \Phi_{1k}^T \ddot{K}_{1d} \end{bmatrix} \]
and
\[ \ddot{B}_{1j}^* = \begin{bmatrix} \ddot{B}_{1j}^* \ddot{f}_{1j}^* \\ \ddot{B}_{1F}^* \ddot{F}_1 \end{bmatrix}, \quad \ddot{B}_{1F}^* = \begin{bmatrix} \ddot{B}_{1F}^* \ddot{B}_{1F}^* \ddot{f}_{1j}^* \ddot{F}_1 \end{bmatrix} \]
Writing Eq. (6) into the block matrix form, we have
\[ \ddot{M}_{1k}^* \ddot{p}_{1k} + \ddot{K}_{1k}^* p_{1k} = \ddot{B}_{1j}^* \ddot{f}_{1j} + \ddot{B}_{1F}^* \ddot{F}_1; \]
and
\[ \ddot{M}_{1d}^* \ddot{p}_{1d} + \ddot{K}_{1d}^* p_{1d} = \ddot{B}_{1j}^* \ddot{f}_{1d} + \ddot{B}_{1F}^* \ddot{F}_1; \]
where Eqs. (7a) and (7b) are the dynamic equations of the high and high modes of Substructure 1, respectively. We know that the response of Substructure 1 is mainly dominated by Eq. (7a), and the contribution of high modes to the response of Substructure 1 is close to the static part. Neglecting dynamic terms and the external force in Eq. (7b), the static equation of Eq. (7b) can be written as
\[ \ddot{K}_{1d}^* p_{1d} = \Phi_{1d}^T \ddot{B}_{1j}^* \ddot{f}_{1j}. \]
Thus,
\[ p_{1d} = \ddot{K}_{1d}^*^{-1} \Phi_{1d}^T \ddot{B}_{1j}^* \ddot{f}_{1j}. \]
Substituting Eq. (9) into Eq. (5), we have
\[ \ddot{u}_1^* = \Phi_{1k} p_{1k} + \ddot{K}_{1d}^* \ddot{K}_{1d}^{-1} \Phi_{1d}^T \ddot{B}_{1j}^* \ddot{f}_{1j} \]
\[ = \Phi_{1k} p_{1k} + \ddot{K}_{1d} \ddot{f}_{1j} \]
where \( \ddot{K}_{1d} = \Phi_{1d}^T \ddot{K}_{1d}^{-1} \Phi_{1d}^T \ddot{B}_{1j}^* \ddot{f}_{1j} \) is actually the residual flexibility of the interface force \( f_{1j} \), and it is called the residual mode; also, it represents the static contribution of high modes. In the double-compatible, free-interface modal synthesis method, \( \ddot{K}_{1d} \) will be used in modal synthesis for the system. However, in the classical free-interface modal synthesis method, only the low modes of Substructure 1 are used in modal synthesis, and \( \ddot{K}_{1d} \) is useless. In Section 4, it will be demonstrated through numerical simulation that the use of \( \ddot{K}_{1d} \) can greatly improve the accuracy of modal synthesis. Equation (10) shows that, just like the modal coordinate \( p_{1k} \), the interface force \( f_{1j} \) can be also regarded as a kind of modal coordinate.

Substituting Eq. (10) into Eq. (4) and left multiplying \( \Phi_{1n}^T \), one can obtain
\[ \ddot{M}_1^* \ddot{p}_1 + \ddot{K}_1^* p_1 = \ddot{B}_j^* f_{1j} + \ddot{B}_{1F} F_1; \]
where
\[ p_1 = [p_{1k}^T \ p_{1d}^T]^T; \]
\[ \ddot{M}_1^* = \begin{bmatrix} \ddot{M}_{1k} & 0 \\ \Phi_{1k}^T \ddot{M}_{1d} \end{bmatrix} \]
and
\[ \ddot{K}_1^* = \begin{bmatrix} \ddot{K}_{1k} & 0 \\ \Phi_{1k}^T \ddot{K}_{1d} \end{bmatrix} \]
and
\[ \ddot{B}_j^* = \begin{bmatrix} \ddot{B}_j^* \ddot{f}_{1j} \\ \ddot{B}_{1F} \ddot{F}_1 \end{bmatrix}, \quad \ddot{B}_{1F} = \begin{bmatrix} \ddot{B}_{1F} \ddot{B}_{1F} \ddot{f}_{1j} \ddot{F}_1 \end{bmatrix} \]
where \( \ddot{M}_{1d}^* \) and \( \ddot{K}_{1d}^* \) are the residual mass and stiffness matrices, respectively.

Equation (11) is the modal representation of Substructure 1 that will be used for the modal synthesis of the system. To make the same treatment for Substructures 2 and 3, two modal equations as Eq. (11) can be obtained, and the processing procedure is omitted herein. Assembling the three modal equations, we have
\[ \dddot{M}^* \dddot{p} + \dddot{K}^* \dddot{p} = \dddot{f}^* + \dddot{B}^* \dddot{F} \]
where
\[ p = [p_{1k}^T \ p_{1d}^T]^T, \quad \dddot{M}^* = \text{diag}(M_{1k}^* M_{2k}^* M_{3k}^*); \]
\[ \dddot{K}^* = \text{diag}(K_{1k}^* K_{2k}^* K_{3k}^*); \]
\[ \dddot{f}^* = \left( \begin{bmatrix} \dddot{B}_{1j}^* \dddot{f}_{1j} \dddot{B}_{2j}^* \dddot{f}_{2j} \dddot{B}_{3j}^* \dddot{f}_{3j} \end{bmatrix} \right)^T; \]
and
\[ \dddot{B}^* = \begin{bmatrix} \dddot{B}_{1F}^* & 0 & 0 \\ 0 & \dddot{B}_{2F}^* & 0 \\ 0 & 0 & \dddot{B}_{3F}^* \end{bmatrix}, \quad \dddot{F} = [F_{1j} F_{2j} F_{3j}]^T. \]

Equation (12) is not the independent coordinate equation of the system since the elements of \( \dddot{p} \) are not independent. The vectors \( p_{1j}, p_{2j}, \) and \( p_{3j} \) all contain the interface DOFs of the system. Below, we apply the displacement and force compatibility conditions of the three substructures to eliminate the redundant DOFs so as to obtain the independent coordinate equation of the system. The displacement compatibility equation of the three substructures is
\[ u_{1j} = u_{2j} = u_{3j}; \]
where \( u_{1j}, u_{2j}, \) and \( u_{3j} \) are the vectors of the interface coordinates of the three substructures, respectively. They can be also written as
\[ u_{1j} = B_1 u_{1i}^*, \quad u_{2j} = B_2 u_{2i}^*, \quad u_{3j} = B_3 u_{3i}^*; \]
where $B_1$, $B_2$, and $B_3$ are the Boole indicated matrices of the three substructures, respectively, and their function is to get the interface displacement vectors $u_{1j}$, $u_{2j}$, and $u_{3j}$ from the entire displacement vectors $u_1^+$, $u_2^+$, and $u_3^+$. Using Eq. (10), we have

$$B_1 (\Phi_{1k} p_{1k} + \Psi_{1d} f_{1j}) = B_2 (\Phi_{2k} p_{2k} + \Psi_{2d} f_{2j});$$  (15)

and

$$B_3 (\Phi_{3k} p_{3k} + \Psi_{3d} f_{3j}) = B_2 (\Phi_{2k} p_{2k} + \Psi_{2d} f_{2j});$$  (16)

where $f_{1j}$, $f_{2j}$, and $f_{3j}$ are the interface-force vectors of the three substructures, respectively. Equations (15) and (16) are the compatibility equations of the interface displacement described using the modal coordinate $p_{ik}$ ($i = 1 - 3$) and the interface force $f_{ij}$ ($i = 1 - 3$). Using Eqs. (15) and (16), we can written as the function of $p_{ik}$; thus, it may be eliminated in the process of modal synthesis. The detailed process is given below:

From Eqs. (15) and (16), one can obtain

$$f_{1j} = (B_1 \Psi_{1d})^{-1} [B_2 (\Phi_{2k} p_{2k} + \Psi_{2d} f_{2j}) - B_1 \Phi_{1k} p_{1k}];$$  (17)

and

$$f_{3j} = (B_3 \Psi_{3d})^{-1} [B_2 (\Phi_{2k} p_{2k} + \Psi_{2d} f_{2j}) - B_3 \Phi_{3k} p_{3k}].$$  (18)

The condition of the interface force of the three substructures is

$$f_{1j} + f_{2j} + f_{3j} = 0.$$  (19)

Substituting Eqs. (17) and (18) into Eq. (19), we have

$$f_{2j} = \Delta_2 \begin{pmatrix} p_{1k} \\ p_{2k} \\ p_{3k} \end{pmatrix};$$  (20)

where

$$\Delta_2 = \Pi_2^{-1} \left[ (B_1 \Psi_{1d})^{-1} B_1 \Phi_{1k} \\
(\Phi_{1d})^{-1} B_2 \Phi_{2k} - (B_3 \Psi_{3d})^{-1} B_3 \Phi_{2k} \\
(\Psi_{3d})^{-1} B_3 \Phi_{3k} \right];$$

$$\Pi_2 = (B_1 \Psi_{1d})^{-1} B_2 \Psi_{2d} + I + (B_3 \Psi_{3d})^{-1} B_2 \Psi_{2k}.$$

Substituting Eq. (20) into Eqs. (17) and (18), we have

$$f_{1j} = \Delta_1 \begin{pmatrix} p_{1k} \\ p_{2k} \\ p_{3k} \end{pmatrix}, \quad f_{3j} = \Delta_3 \begin{pmatrix} p_{1k} \\ p_{2k} \\ p_{3k} \end{pmatrix};$$  (21)

where

$$\Delta_1 = \left[ - (B_1 \Psi_{1d})^{-1} B_1 \Phi_{1k} \quad (B_1 \Psi_{1d})^{-1} B_2 \Phi_{2k} \quad 0 \right]$$

$$+ (B_1 \Psi_{1d})^{-1} B_2 \Psi_{2d} \Delta_2;$$

and

$$\Delta_3 = \left[ 0 \quad (B_3 \Psi_{3d})^{-1} B_2 \Phi_{2k} - (B_3 \Psi_{3d})^{-1} B_3 \Phi_{3k} \right]$$

$$+ (B_3 \Psi_{3d})^{-1} B_2 \Psi_{2d} \Delta_2.$$

Define the independent coordinate of the system as

$$q = \begin{pmatrix} p_{1k} \\ p_{2k} \\ p_{3k} \end{pmatrix}.\quad (22)$$

The relationship between $q$ and $p$ in Eq. (12) is

$$p = \begin{pmatrix} p_{1k} \\ f_{1j} \\ p_{2k} \\ f_{2j} \\ p_{3k} \\ f_{3j} \end{pmatrix} = S \tilde{q} = S \begin{pmatrix} p_{1k} \\ p_{2k} \\ p_{3k} \end{pmatrix};$$  (23)

where $S$ is the coordinate transformation matrix that can be obtained by the matrix assembly using Eqs. (20) and (21). Substituting Eq. (23) into Eq. (12) and left multiplying $S^T$, we have

$$Mq + Kq = \tilde{B} \tilde{F};$$  (24)

where $M = S^T M S$, $K = S^T K S$, and $B = S^T \tilde{B}$. Since the interface force appears in pairs in the structural system, only the external force $F$ appears in the right hand of Eq. (24). Equation (24) is the final dynamic equation of the system obtained using the double-compatible modal synthesis method, which can be used for dynamic analysis and control design for the system. In this equation, all the interface forces are eliminated and only the low-order modal coordinates of each component of the system are preserved, so the order dimension of Eq. (24) is the amount sum of the preserved low-order modes of every component of the system.

### 3. INTEGRAL REDUCTION USING THE MODAL COST ANALYSIS METHOD

With considering the observation equation, the dynamic system can be written as

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} \tilde{C}_d \dot{q} + \tilde{C}_e q + \tilde{C}_w \end{pmatrix} = \begin{pmatrix} \tilde{C}_d \dot{q} + \tilde{C}_e q + \tilde{C}_w \end{pmatrix} \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix};$$  (25)

where $y$ is the output vector of the system. $\tilde{C}_d$ and $\tilde{C}_e$ are the observation matrices of displacement and velocity, respectively. The motion described by Eq. (25) contains rigid motion and flexible vibration of the system. Rigid motion refers to the large motion of the system and flexible vibration refers to the elastic vibration of the system in the body-fixed frame. Actually, model reduction for the system (25) means the order reduction of the part of flexible vibration, and rigid motion does not need any order reduction. Below the system (25) is firstly decoupled into rigid motion and flexible vibration, and then the modal cost analysis method will be used for order reduction of the flexible vibration.

The system (25) is positive semi-definite vibration system, so $M$ is a positive definite matrix and $K$ a positive semi-definite one. Based on the theory of vibration, there exists a nonsingular transformation $q = T\tilde{q}$ for the system such that

$$\begin{pmatrix} T^T MT = I, \quad T^T KT = diag(0, \omega_j^2) \end{pmatrix};$$  (26)

where $T$ is the modal transformation matrix and $\omega_j^2 = diag(\omega_j^2)$ is a diagonal matrix with its element being natural frequency of flexible vibration of the system. Dividing $T$
into the rigid-mode part \( T_g \) and the flexible-vibration part \( T_e \), and also correspondingly dividing \( \hat{\eta} \) into two parts, both are given by

\[
T = [T_g \ T_e], \quad \hat{\eta} = [\eta_g^T \ \eta_e^T]^T.
\]

(27)

So, the system (25) can be decoupled into two parts as

\[
\begin{align*}
\hat{\eta}_g &= T_g^T BF \\
y_g &= C_d T_g \eta_g + C_r T_g \hat{\eta}_g
\end{align*}
\]

(28)

and

\[
\begin{align*}
\hat{\eta}_e &= \omega_e^2 \eta_e + T_e^T BF \\
y_e &= C_d T_e \eta_e + C_r T_e \eta_e
\end{align*}
\]

(29)

Equations (28) and (29) are the equations of rigid motion and flexible vibration of the system, respectively.

With considering the damping effect, the flexible vibration equation can be written as

\[
\begin{align*}
\hat{\eta}_e + \text{diag}(2\xi_i \omega_i) \hat{\eta}_e + \text{diag}(\omega_e^2) \eta_e &= BF \\
y_e &= C_d \eta_e + C_r \eta_e
\end{align*}
\]

(30)

where \( C_d = C_d T_e, C_r = C_r T_e \) and \( B = T_e^T B \); the parameters \( \omega_i \) and \( \xi_i \) are the \( i \)-th modal natural frequency and damping ratio, respectively.

In the state space representation, Eq. (30) becomes

\[
\begin{align*}
\dot{x} &= Ax + BF \\
y &= Cx
\end{align*}
\]

(31)

where

\[
x = [\hat{\eta}_e, \omega_1 \eta_{e1}, \ldots, \omega_n \eta_{en}]^T; \quad A = \text{diag}(A_i), \quad A_i = \begin{bmatrix} 2\xi_i \omega_i & -\omega_i & 0 \\ -\omega_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad B = [B_{i1}^T, \ldots, B_{i1}^T]^T, \quad B_i = \begin{bmatrix} b_{i1} \\ 0 \end{bmatrix}, \quad b_{i1} \text{ is the } i \text{-th row of } \hat{B}; \quad C = [C_1, \ldots, C_n], \quad C_i = \begin{bmatrix} C_{r1}, C_{d1}/\omega_1 \end{bmatrix}, \quad C_{ri} \text{ and } C_{di} \text{ are the } i \text{-th column of } C_r \text{ and } C_d, \text{ respectively.}
\]

The state component of Eq. (31) is the modal coordinate, so the cost of component is called the modal cost. The modal cost of \( i \)-th mode of the system is

\[
V_i = tr(\hat{X} C_i^T Q C_i^T)_{n}.
\]

(32)

where \( tr(\bullet) \) represents the matrix trace, \( Q \) is a weight matrix with its elements denoting the important degree of every component of the output vector \( y \), and \( \hat{X} \) is the solution of the following Lyapunov equation.\(^{14-17} \) The equation is listed below:

\[
XA^T + AX + BB^T = 0.
\]

(33)

The total modal cost \( V \) of the system is the sum of every modal cost, given by

\[
V = \sum_{i=1}^{n} V_i.
\]

(34)

where \( n \) is the number of the total modes of the system.

Skelton and Hughes ever introduced a modal quality index (MQI), given by\(^{16} \)

\[
MQI = \left( \sum_{i=1}^{r} V_i \right) / \left( \sum_{i=1}^{n} V_i \right) = \left( \sum_{i=1}^{r} V_i \right) / V; \quad (35)
\]

\[\text{where } \left( \sum_{i=1}^{r} V_i \right) \text{ is the sum of modal cost of the preserved modes of the system and } r \text{ is the number of the preserved modes.}\]

The preserved modes have large values of modal cost and the reduction model of flexible vibration of the system is composed of those preserved mode. The more the value of MQI is close to 1, the better the quality of the reduction model is. It should be mentioned herein that the preserved mode mentioned herein means the one having large value of modal cost calculating using Eq. (32), and it has different meaning from the mode mentioned in the modal synthesis technique in Section 2.

### 4. NUMERICAL SIMULATIONS

In this section, numerical simulations are carried out to demonstrate the validity of the proposed model reduction method in this paper. In Fig. 2, the length of long beam is \( l_2 = 55 \) m, the cross section area is \( A_2 = 1.2177 \) m², the modulus of elasticity is \( E_2 = 72 \) GPa and the mass is \( m_{12} = 1.8150 \times 10^5 \) kg. The parameters of the two short beams are: \( l_3 = l_4 = 15 \) m, \( A_3 = A_4 = 0.5454 \) m², \( E_3 = E_4 = 72 \) GPa and \( m_1 = m_2 = 2.2171 \times 10^4 \) kg.

Firstly consider the local reduction of the system, namely the modal synthesis method is applied for the system. In using the FEM, the long beam is divided into 55 elements and the short beam 15 elements, so there are 56 nodes and 168 DOFs for the long beam, and 16 nodes and 48 DOFs for the short beam. The total number of DOFs of the system using the FEM is 264. In using the modal truncation method for the three substructures, the top 1/3 of modes of the long beam and the first ten modes of the two short beams are preserved, so the total number of the preserved modes of the system is 76. Therefore, the order of the reduction model established using the double-compatible modal synthesis method is 76. We can see that the model order of the system has been reduced greatly by the first step of reduction. To verify the validity of the reduction model by the first step, natural frequencies of the system obtained using the FEM and the double-compatible modal synthesis method are compared numerically. Table 1 shows the results of the first thirty frequencies, where the results using the FEM are given in the second column and those using the double-compatible modal synthesis method in the fifth column. The results using the classical free-interface and fixed-interface modal synthesis methods are shown in the third and fourth columns, respectively, for comparison too. The first three frequencies in Tab. 1 are zero, representing the rigid modes of the system. Here we display the frequency error using the frequency ratio \( \frac{\omega_i}{\omega_0} \), where \( \omega_0 \) represents the natural frequency obtained using the FEM, and \( \omega \) using the free-interface, fixed-interface and double-compatible modal synthesis methods, separately. The frequency error is shown in Fig. 3, where Figs. 3b–d are the error curve between the models of the FEM and three modal synthesis methods respectively. In Fig. 3, the solid line is the result using the FEM, the dashed line the free-interface modal synthesis method, the small dotted line the fixed-interface modal synthesis method and the big dotted line the double-compatible modal synthesis method.

Table 1 lists the natural frequencies of the system using the FEM method and the three modal synthesis methods. Since
Figure 3: Frequency error of the system

Table 1: Natural frequencies of the system (Hz)

<table>
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<tr>
<th>Order of mode</th>
<th>Finite element modal synthesis method</th>
<th>Free-interface modal synthesis method</th>
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both preserved modes and residual modes are both considered in using the double-compatible free-interface method, this method can achieve almost the same result as the FEM method. The free-interface method may gain a significant advantage as model complexity increases. With this method, fewer degrees of freedom are needed to attain the same level of accuracy as the fixed-interface method, thus the free-interface method gets more applications than the fixed interface method. But the free-interface and fixed-interface methods both use the preserved modes only, so the accuracy using these two methods are lower than the double-compatible free-interface method.

Then consider the integral reduction of the system, namely the modal cost analysis method is used for the system. All the modal damping ratio of the system are chosen as 0.005. The three external forces acts at the bottom of the three beam, and the tip displacement of the long beam is assumed to be observed and no velocity is observed. Thus the position denoting matrices B and C in Eq. (31) can be determined. The weighting matrix Q in Eq. (32) is chosen as unit matrix. So the modal cost of every mode of the system can be calculated. Table 2 gives the results of modal cost of the first thirty modes of the system. From Table 2 we can observe that the values of the 4-th and 5-th modes are evidently larger than the others, and the sum of modal cost of these two modes accounts for 99.24% of the total cost of the system, so these two modes are chosen to form a reduction model to represent the flexible vibration of the system approximately.

Next we verify the validity of the reduction model obtained by the integral reduction. The frequency response of flexible vibration of the system is used as the judgment standard to evaluate the effectiveness of dynamic model. Three models are used in the simulations: (i) the FEM model of the system; (ii) the reduction model with 76 DOFs obtained by the first step of reduction; (iii) the reduction model with 2 DOFs obtained by the two steps of reduction. The simulation results are shown in Figs. 4–6, where Fig. 4 is the result using the FEM model, Fig. 5 the model by the first step of reduction and Fig. 6 by the two steps of reduction. We can observe from Figs. 4–6 that the results using the two reduction models agree better with that using the FEM model, indicating that the two reduction model can reflect the dynamic characteristics of the original system effectively. However, the model order by the two-step reduction is greatly smaller than that by the first-step reduction. So it will be more easy and convenient using the two-step reduction model for dynamic analysis and control design for the system.

5. CONCLUDING REMARKS

In this paper, model reduction is studied using a strap-on launch vehicle as research object and a two-step reduction model
method is proposed. In this method, the first step of reduction is done using the double-compatible modal synthesis method to establish the dynamic model of the system, then the modal cost analysis method is used as the second step of reduction to reduce the order of the dynamic equation established. In the numerical simulations, the calculating precision of the double-compatible modal synthesis method is compared with that of the classical free-interface and fixed-interface modal synthesis methods. Simulation results indicate that the calculating precision of the double-compatible modal synthesis method is higher than that of the classical free-interface and fixed-interface modal synthesis methods, especially on the high-order modes of the system. The reduction model obtained using the proposed two-step method is effective in describing the original system and has the merit of low order as well.

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