Nonlinear Vibration Analysis of Flexible Hoisting Rope with Time-Varying Length

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(Received 3 March 2013; accepted 14 January 2015)

The nonlinear vibration of a flexible hoisting rope with time-varying length and axial velocity is investigated. The flexible hoisting rope is modeled as a taut translating string with a rigid body attached at its low end. A systematic procedure for deriving the system model of a flexible hoisting rope with time-varying length and axial velocity is presented. The governing equations were developed by employing the extended Hamilton's principle considering coupling of axial movement and flexural deformation of the rope. The derived governing equations are nonlinear partial differential equations(PDEs) with time-varying coefficients. The Galerkin's method and the 4^{th} Runge-Kutta method were employed to numerically analyze the resulting equations. Further, the dynamic stability of the flexible hoisting rope was investigated according to the Lyapunov stability theory. The motions of an elevator hoisting system were presented to illustrate the proposed mathematical models. The results of simulation show that the dynamic motions of the flexible hoisting string are stable during downward movement but are unstable during upward movement. The proposed systematic procedures in analyzing the dynamic stability can facilitate further development in dynamic control of the flexible hoisting system in practice.

NOMENCLATURE

a	Axial acceleration of the string (m/s^2)
$\mathbf{A}$, $\mathbf{B}$	Matrix differential operators
$\mathbf{C}$	Damp matrixes
d	Diameter of the string (m)
E	Young's Modulus of the string (Pa)
E_k	Kinetic energy of flexible hoisting system (J)
E_e	Elastic strain energy of the string (J)
E_g	Gravitational potential energy of flexible hoisting system (J)
g	Gravitational constant (m/s^2)
$\mathbf{i}$	Unit vector along the x-axes
i	Integer
I	Inertia (m^4)
$\mathbf{I}$	$n \times n$ identity matrix
$\mathbf{j}$	Unit vector along the y-axes
j	Integer
k	Integer
$\mathbf{K}$	Stiffness matrixes
l	Length of the string (m)
m	Mass of rigid body (kg)
$\mathbf{M}$	Mass matrixes
n	Number of included modes
P	Longitudinal tension (N)
q_i	Generalized coordinates
$\mathbf{Q}$	Vectors of generalized coordinates
$\mathbf{R}$	Position vector of the string
$\mathbf{R_c}$	Position vector of the rigid body
t	Time (s)
T	Lyapunov candidate function
U	State vector
v	Axial velocity of the string (m/s)
$\mathbf{V}$	Velocity vector of the string
$\mathbf{V_c}$	Velocity vector of the rigid body
x	Spatial variable (m)
y	Transverse displacement of the string (m)
ζ	Transformed spatial variable
ε	Strain measure
ρ	Linear density of the string (kg/m)
λ_k	Eigenvalue ($k = 1, 2, 3, 4$)
ξ_k	Real parts of eigenvalue ($k = 1, 2, 3, 4$)
φ_i	Trial function used in Eq. (19)
δ_{ij}	Kronecker delta
ω_k	Imaginary parts of eigenvalue ($k = 1, 2, 3, 4$)
$\mathbf{\Lambda}$	Eigenvector

1. INTRODUCTION

Ropes with time-varying length are widely used in the hoisting industry such as mine hoists, elevators, cranes, etc; They are subject to vibration due to their high flexibility and relatively low internal damping characteristics.[1,2] Most often these systems are modeled as either an axially moving tensioned beam or as a string with time-varying length and a rigid body at its lower end.[3,4] It was reported that the vibration energy of the rope changed in general during elongation and shortening.[5–7] Zhang[8–11] and Bao[12,13] published a series of studies on vibration of a flexible hoisting system with arbitrarily varying length. Terumichi et al. assumed the velocity of the string was constant and studied the transverse vibrations of a string with time-varying length and a mass-spring system at the lower end with theoretical and experimental methods.[14] Zhu[15] and Chen[16] investigated the control of an elevator cable with theoretical and experimental methods. A novel experimental method was developed to validate the uncontrolled and controlled lateral responses of a moving cable in a high-rise elevator and showed good agreement with the theoretical predictions. Nguyen and Hong studied the transverse vibration control of axially moving membranes by regulation of axial velocity.[17] A novel control algorithm that suppresses the transverse vibrations of an axially moving membrane system was developed. Ngo et al. investigated the control of an axially moving system. The Lyapunov function taking the form of the total mechanical energy of the system was adopted to ensure the uniform stability of the closed-loop system.[18] The results of experiments showed that the proposed control law was effective. Fung and Lin analyzed the transverse vibration of an elevator rope with time-varying length and the time-varying mass and inertia of rotors were considered.[19] A variable structure control scheme was proposed to suppress the transient amplitudes of vibrations. Chi and Shu calculated the natural frequencies associated with the vertical vibration of a stationary cable coupled with an elevator car.[20] Zhang presented a systematic procedure for deriving the model of a cable transporter system with arbitrarily varying cable length and proposed a Lyapunov controller to dissipate the vibratory energy.[21] Kaczmarczyk and Ostachowicz studied coupled vibration of a deep mine hoisting cable and built a distributed-parameter model. They found that the response of the catenary-vertical rope system may feature a number of resonance phenomena.[22] Zhang and Agrawal derived the governing equation of coupled vibration of a flexible cable transporter system with arbitrarily varying length.[23]

While an extensive number of studies focus individually on vibration characteristics of the rope with time-varying length, the dynamic stability of the rope has also been studied by several researchers. Lee introduced a new technique to analyze free vibration of a string with time-varying length by dealing with traveling waves.[24] As the string length is shortened, free vibration energy increases exponentially, causing dynamic instability. Kumaniecka and Niziol investigated the longitudinal-transverse vibration of a hoisting cable with slow variability of the parameters.[25] The cable material non-linearity was taken into account and unstable regions were identified by applying the harmonic balance method. General stability characteristics of horizontally and vertically translating beams and strings with arbitrarily varying length and various boundary conditions were studied by Zhu and Ni.[26] While the amplitude of the displacement can behave in a different manner depending on the boundary conditions, the amplitude of the vibratory energy of a translating medium decreases and increases during extension and retraction, respectively.

Extensive research on the flexible hoisting rope with time-varying length has been conducted in the last few decades as aforementioned; however, the focus of most studies was restricted to cases with constant transport speed samples. Clearly, with the advancement of high-performance mechanical systems such as high-rise elevators, cranes and mine hoists, etc., the stability analysis of dynamical systems is very important. The linear dynamic characteristics and stability of the flexible hoisting rope with time-varying length and axial velocity are the subject of this investigation. The governing equations were developed employing the extended Hamilton's principle. The derived governing equations are shown to be linear partial differential equations (PDEs) with variable coefficients. On choosing proper mode functions that satisfy the boundary conditions, the solution of the governing equations was obtained using the Galerkin's method. The natural frequencies were computed from the eigenvalues based on the eigenvalue equations. The motions of an elevator hoisting system were illustrated to evaluate the proposed mathematical models. According to the numerical simulations, the dynamic motions of the flexible hoisting string are stable during downward movement but are unstable during upward movement. Based on the proposed fundamental dynamic analyses, further vibration control can be adopted for flexible hoisting systems in the near future.

2. MODEL OF FLEXIBLE HOISTING SYSTEM

A flexible hoisting system is simplify an axially moving string with time-varying length and a rigid body m at its lower end, as shown in Fig. 1. The rail and the suspension of the rail are assumed to be rigid. The string has Young's modulus E, diameter d and mass per unit length ρ. The origin of the coordinate is set at the top end of the string, and the instantaneous length of the string is $l(t)$ at time t. The instantaneous axial velocity, acceleration and jerk of the string are $v(t) = \dot{l}(t)$, $a(t) = \dot{v}(t)$, and $j(t) = \dot{a}(t)$, respectively, where the overdot denotes time differentiation. At any instant t, the transverse displacement of the string is described by $y(x, t)$, at a spatial position x, where $0 \leq x \leq l(t)$. The model is based on the following assumptions:

1. The parameters E, d and ρ of the string are always constants;
2. Only transverse vibration is considered here. The elastic distortion of the string arousing from the transverse vibration is much less than the length of the string;
3. All the damp and friction, and the influence of air currents are ignored.

2.1. Energy of Flexible Hoisting System

After the string is deformed, the position vector $\mathbf{R}$ of a point at x can be written as:[27]

$$\mathbf{R} = x(t)\,\mathbf{i} + y(x, t)\,\mathbf{j}; \tag{1}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors along the x-axes and y-axes, respectively. The material derivative of $\mathbf{R}$ yields the velocity vector

$$\mathbf{V} = v(t)\,\mathbf{i} + [y_t + v y_x]\,\mathbf{j}; \tag{2}$$

where the subscript t denotes partial differentiation with respect to time, and subscript x denotes partial differentiation with respect to space. Similarly, the position vector $\mathbf{R_c}$ and

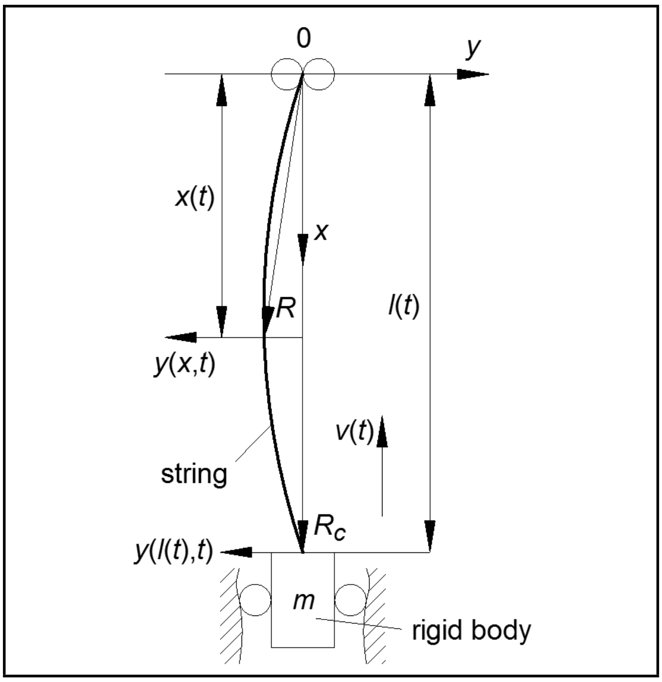

Figure 1. Schematic of a flexible hoisting string with time-varying length.

velocity vector $\mathbf{V_c}$ of the rigid body can be respectively written as:

$$\mathbf{R_c} = l(t)\,\mathbf{i} + y(l(t),t)\,\mathbf{j}; \tag{3}$$

$$\mathbf{V_c} = v(t)\,\mathbf{i} + y_t(l(t),t)\,\mathbf{j}. \tag{4}$$

Then, the kinetic energy of the flexible hoisting system is computed by

$$E_k(t) = \frac{1}{2}m\mathbf{V_c}\cdot\mathbf{V_c}\Big|_{x=l(t)} + \frac{1}{2}\rho\int_0^{l(t)}\mathbf{V}\cdot\mathbf{V}\,dx. \tag{5}$$

The first term on the right of Eq. (5) represents the kinetic energy of the rigid body; The second term represents the kinetic energy of the string. The elastic strain energy of the string is[28]

$$E_e(t) = \int_0^{l(t)}\left(P\varepsilon + \frac{1}{2}EA\varepsilon^2\right)\,dx. \tag{6}$$

The first term on the right of Eq. (6) represents the axial strain energy of the string, the second term represents the bending strain energy of the string. $P(x,t)$ is the longitudinal tension at spatial position x of the string at time t; The tension in a flexible hoisting string, arising from its weigh, is given by

$$P = [m + \rho\,(l(t) - x)]\,g; \tag{7}$$

and ε represents the strain measure at the spatial position x of the string and can be expressed as

$$\varepsilon = (ds - dx)/dx. \tag{8}$$

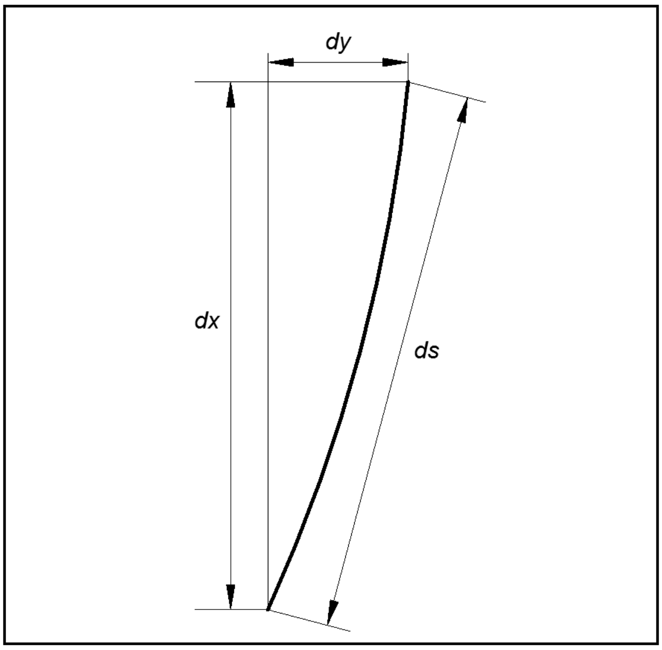

Figure 2. A small element of the string in a deformed position.

As shown in Fig. 2, ds can be expressed as

$$ds \approx \sqrt{1 + (dy/dx)^2}dx \approx \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial y}{\partial x}\right)^4 + \ldots\right]dx \approx \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]dx. \tag{9}$$

Substituting Eq. (9) into Eq. (8) yields

$$\varepsilon = \frac{1}{2}y_x^2. \tag{10}$$

When the reference elevation of the string with zero potential energy is defined at $x = 0$, then the gravitational potential energy of the flexible hoisting system is

$$E_g(t) = -\int_0^{l(t)}\rho g x(t)\,dx - mgl(t); \tag{11}$$

where g is the gravitational constant, the first term on the right of Eq. (11) represents the gravitational potential energy of the string, and the second term represents the gravitational potential energy of the rigid body.

2.2. Governing Equations of Motion

According to the characteristics of top restriction of the string, the boundary conditions at $x(t) = 0$ are

$$y(0,t) = 0, \qquad y_t(0,t) = 0. \tag{12}$$

Substitute Eqs. (5), (6), and (11) in the extended Hamilton's Principle,

$$\int_{t_1}^{t_2}\Big(\delta E_k(t) - \delta E_e(t) - \delta E_g(t)\Big)dt = 0; \tag{13}$$

and apply the variational operation. Because the length of the string $l(t)$ changes with time, the standard procedure for integration by parts with respect to the temporal variable cannot apply. Applying Leibnitz's rule and part integration results in the following expressions

$$\int_0^{l(t)} \rho\,(y_t + v y_x)\,\delta y_t\,dx = \rho\frac{\partial}{\partial t}\int_0^{l(t)} (y_t + v y_x)\,\delta y\,dx - \rho\big[v\,(y_t + v y_x)\,\delta y\big]\Big|_{l(t)} - \rho\int_0^{l(t)} \frac{\partial}{\partial t}\,(y_t + v y_x)\,\delta y\,dx. \quad (14)$$

Following the standard procedure for integration by parts with respect to the spatial variable and invoking Eq. (14), one obtains from Eq. (13),

$$\int_{t_1}^{t_2}\left[m\frac{\partial}{\partial t}y_t(l,t) + \Big(P + \frac{1}{2}EAy_x^2\Big)y_x\right]\Big|_{l(t)}\delta y(l,t)\,dt + \int_{t_1}^{t_2}\int_0^{l(t)}\left[\rho\frac{\partial}{\partial t}\,(y_t + v y_x) + \rho v\frac{\partial}{\partial x}\,(y_t + v y_x) - \frac{\partial}{\partial x}\left(\Big(P + \frac{1}{2}EAy_x^2\Big)y_x\right)\right]\delta y\,dx\,dt = 0. \quad (15)$$

Setting the coefficients of δy in Eq. (15) to zero yields the governing equation for the string,

$$\rho\left(y_{tt} + 2\dot{v}y_x + v^2 y_{xx}\right) - P_x y_x - P y_{xx} - \frac{3}{2}EAy_x^2 y_{xx} = 0, \quad 0 < x < l(t). \quad (16)$$

The first four terms in Eq. (16) correspond to the local, Coriolis, tangential, and centripetal acceleration, respectively. Equation (16) is a partial differential equation that describes the dynamics of the flexible hoisting string. The equation is defined over time-dependent spatial domain rendering the problem non-stationary. Hence, the exact solution to this problem is not available, and recourse must be made to an approximate analysis. In what follows, numerical techniques are employed to obtain an approximate solution for the governing equation.

3. DISCRETIZATION OF THE GOVERNING EQUATIONS

Equation (16) is a partial differential equation with infinite dimensions and many parameters are time-variant. It is impossible to obtain an exact analytical solution from Eq. (16). In this section, Galerkin's method was applied to truncate the infinite-dimensional partial differential equation into a linear finite-dimensional ordinary differential equation with time-variant coefficients. Then, they were solved with numerical methods. In order to map Eq. (16) onto the fixed domain, a new independent variable $\zeta = x/[l(t)]$ was introduced and the time-variant domain $[0, l(t)]$ for x was converted to a fixed domain $[0, 1]$ for ζ. According to the characteristic of a taut translating string, the solution of $y(x,t)$ was assumed in the forms[15,23]

$$y(x,t) = \sum_{i=1}^{n}\varphi_i(\zeta)q_i(t) = \sum_{i=1}^{n}\varphi_i\left(\frac{x}{l}\right)q_i(t); \quad (17)$$

where $q_i(t)$ $(i = 1, 2, 3, \ldots, n)$ is the generalized coordinate respect to $y(x,t)$, n is the number of included mode, and $\varphi_i(\zeta)$ is trial function,[15,23]

$$\varphi_i(\zeta) = \sqrt{2}\,\sin i\pi\zeta. \quad (18)$$

Consequently, expanding Eq. (17) results in the expressions for partial derivatives of transverse displacement functions:

$$y_x(x,t) = \frac{1}{l}\sum_{i=1}^{n}\varphi_i'(\zeta)q_i(t),$$
$$y_{xx}(x,t) = \frac{1}{l^2}\sum_{i=1}^{n}\varphi_i''(\zeta)q_i(t),$$
$$y_{xt}(x,t) = \sum_{i=1}^{n}\frac{1}{l}\varphi_i'(\zeta)\dot{q}_i(t) - \sum_{i=1}^{n}\left(\frac{\zeta v}{l^2}\varphi_i''(\zeta) + \frac{v}{l^2}\varphi_i'(\zeta)\right)q_i(t),$$
$$y_{tt}(x,t) = \sum_{i=1}^{n}\varphi_i(\zeta)\ddot{q}_i(t) - \frac{2\zeta v}{l}\sum_{i=1}^{n}\varphi_i'(\zeta)\dot{q}_i(t) + \sum_{i=1}^{n}\left(\frac{2\zeta v^2}{l^2}\varphi_i'(\zeta) - \frac{\zeta a}{l}\varphi_i'(\zeta) + \frac{\zeta^2 v^2}{l^2}\varphi_i''(\zeta)\right)q_i(t). \quad (19)$$

Substituting Eqs. (17)–(19) into Eq. (16), multiplying the governing equation by $\varphi_j(\zeta)$ $(j = 1, 2, 3, \ldots, n)$, integrating it from $\zeta = 0$ to 1, and using the boundary conditions and the orthonormality relation for $\varphi_i(\zeta)$, yield the discretized equation of transverse vibration for the flexible hoisting string with time-variant coefficients

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{C}\dot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} + \mathbf{S}(\mathbf{Q}) = \mathbf{0}; \quad (20)$$

where $\mathbf{Q} = [q_1(t), q_2(t), \ldots, q_n(t)]^T$ is a vector of the generalized coordinate, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are matrices of mass, dampness and stiffness with respect to $\mathbf{Q}$, respectively. $\mathbf{S}(\mathbf{Q})$ is a higher-order item of the generalized coordinate. The matrices are expressed as follows:

$$M_{ij} = \rho\delta_{ij}, \qquad C_{ij}(t) = \frac{2\rho v}{l}\int_0^1 (1-\zeta)\,\varphi_i'(\zeta)\varphi_j(\zeta)\,d\zeta,$$
$$K_{ij}(t) = \frac{\rho a}{l}\int_0^1 (1-\zeta)\,\varphi_i'(\zeta)\varphi_j(\zeta)\,d\zeta - \frac{\rho v^2}{l^2}\int_0^1 (1-\zeta)^2\,\varphi_i'(\zeta)\varphi_j'(\zeta)\,d\zeta + \frac{\rho g}{l}\int_0^1 (1-\zeta)\,\varphi_i'(\zeta)\varphi_j'(\zeta)\,d\zeta - \frac{mg}{l^2}\int_0^1 \varphi_i''(\zeta)\varphi_j(\zeta)\,d\zeta,$$
$$S_j(\mathbf{Q}) = -\frac{3EA}{2l^4}\int_0^1 \Big(\sum_{i=1}^{n}\varphi_i'(\zeta)q_i(t)\Big)^2\sum_{i=1}^{n}\varphi_i''(\zeta)q_i(t)\varphi_j(\zeta)\,d\zeta; \quad (21)$$

where the superscript " ′ " denotes partial differentiation for the normalized variable ζ, and δ_{ij} is the Kronecker delta defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ ($i = 1, 2, 3, \ldots, n$, $j = 1, 2, 3, \ldots, n$). If the initial displacement and velocity of the string are given by $y(x, 0)$ and $y_t(x, 0)$, respectively, where $0 < x < l(0)$, the initial conditions for the generalized coordinate can be obtained from Eqs. (17) and (19);

$$q_i(0) = \int_0^1 y\left(\zeta l(0), 0\right) \varphi_i(\zeta)\, d\zeta; \tag{22}$$

$$\dot{q}_i(0) = \int_0^1 y_t\left(\zeta l(0), 0\right) \varphi_i(\zeta)\, d\zeta + \frac{v(0)}{l(t)} \sum_{i=1}^{n} q_i(0) \int_0^1 \zeta \varphi_i'(\zeta) \varphi_j(\zeta)\, d\zeta. \tag{23}$$

Solving the ordinary differential Eq. (20) with numerical methods may yield the instantaneous values of $\mathbf{Q}$. Substituting these values into Eq. (17) may yield the instantaneous values of transverse vibration of the string $y(x, t)$. The mathematical model defined by Eq. (20) illustrates the true dynamic nature of the flexible hoisting string, and can be used to predict and analyze the dynamic stability and vibration characteristics of a flexible hoisting string with time-varying length and axial velocity.

4. ANALYSIS OF DYNAMIC STABILITY

In order to gain a deeper insight into the mechanics of the flexible hoisting string with time-varying length and axial velocity, it is beneficial to investigate the stability characteristics of the problem. In what follows, we performed a stability analysis of the flexible hoisting string with time-varying dynamic parameters. According Lyapunov's first method, the stability of the system could be determined by analyzing the eigenvalues of the natural vibration. To obtain the eigenvalues of the flexible hoisting string, the methods suggested by Stylianou were used to reduce the system of governing Eq. (20) to a set of first order differential equations.[29] The set of reduced equations takes the form

$$\mathbf{A}\dot{\mathbf{U}} + \mathbf{B}\mathbf{U} = \mathbf{0}; \tag{24}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrix differential operators, and

$$\mathbf{A} = \begin{Bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{Bmatrix}, \qquad \mathbf{B} = \begin{Bmatrix} \mathbf{C} & \mathbf{K} \\ -\mathbf{K} & \mathbf{0} \end{Bmatrix}; \tag{25}$$

$\mathbf{U}$ is the state vector, and

$$\mathbf{U} = \begin{Bmatrix} \dot{\mathbf{Q}} \\ \mathbf{Q} \end{Bmatrix}. \tag{26}$$

Equation (24) is the canonical form of the equation of motion and its solution satisfies the appropriate boundary conditions and initial conditions. Rearranging Eq. (24), we write

$$\dot{\mathbf{U}} + \mathbf{D}\mathbf{U} = \mathbf{0}; \tag{27}$$

where

$$\mathbf{D} = \mathbf{A}^{-1}\mathbf{B} = \begin{Bmatrix} \mathbf{M}^{-1}\mathbf{C} & \mathbf{M}^{-1}\mathbf{K} \\ -\mathbf{I} & \mathbf{0} \end{Bmatrix}. \tag{28}$$

Here, $\mathbf{I}$ is an $n \times n$ identity matrix. To obtain the natural frequencies and mode shapes for the flexible hoisting string with time-varying length, consider the eigenvalue problem of Eq. (27). We now assume that $\mathbf{U}$ is periodic,

$$\mathbf{U} = \mathbf{\Lambda} e^{\boldsymbol{\lambda} t}; \tag{29}$$

where

$$\boldsymbol{\lambda} = \boldsymbol{\xi} + i\boldsymbol{\omega} \tag{30}$$

is the eigenvalue which is a complex number, $\boldsymbol{\xi} = [\xi_1(t), \xi_2(t), \ldots, \xi_k(t)]^T$, $\boldsymbol{\omega} = [\omega_1(t), \omega_2(t), \ldots, \omega_k(t)]^T$ are the real and imaginary parts of $\boldsymbol{\lambda} = [\lambda_1(t), \lambda_2(t), \ldots, \lambda_k(t)]^T$, and $\boldsymbol{\omega}$ is also the natural frequency of the flexible hoisting string. It should be noted that the real and imaginary parts of the eigenvalue are related to the modal damping coefficients and the natrual frequencies of the flexible hoisting string. Substituting Eq. (29) into Eq. (27) leads to an eigenvalue equation

$$(\boldsymbol{\lambda}\mathbf{I} + \mathbf{D})\, \mathbf{\Lambda} = \mathbf{0}; \tag{31}$$

where $\mathbf{\Lambda}$ is the corresponding eigenvector. The eigenvalues can be obtained from

$$\det\left(\boldsymbol{\lambda}\mathbf{I} + \mathbf{D}\right) = 0. \tag{32}$$

When $\boldsymbol{\xi} \leq \mathbf{0}$, the flexible hoisting string is stable, and when $\boldsymbol{\xi} > \mathbf{0}$, the flexible hoisting string is unstable, a positive $\boldsymbol{\xi}$ indicates the instability of the system. The system may lose stability by either divergence (a static form of instability) or flutter (a dynamic form of instability).

The same conclusions can be reached from Lyapunov's second method, which is a mathematical interpretation of the physical property that if a system's total energy is dissipating, then the states of the system will ultimately travel to an equilibrium point. This property can be explored by constructing a scalar, energy-related time-dependent function $T(t)$ for the system, where usually this function $T(t)$ is always positive. If its time derivative $dT(t)/dt < 0$, then the total energy of the system reduces, therefore leading to a stabilized dynamic response. By contrast, if its time derivative $dT(t)/dt > 0$, the total energy of the system increases, thus resulting in an unstabilized dynamic response. So the choice of the appropriate Lyapunov candidate function $T(t)$ is very important. From Eqs. (5) and (6), we observed that the total energy associated with the transverse vibration of the flexible hoisting system is always positive. Hence, the Lyapunov candidate function is given as

$$T(t) = \frac{1}{2} m y_t^2(l(t), t) + \frac{1}{2}\rho \int_0^{l(t)} (y_t + v y_x)^2\, dx + \frac{1}{2} \int_0^{l(t)} \left(P y_x^2 + \frac{1}{4} EA y_x^4\right)^2 dx; \tag{33}$$

where $P(x, t)$ and $y(x, t)$ have been previously defined in Eqs. (7) and (17), respectively. Obviously, $T(t)$ is always positive. Differentiating $T(t)$ in Eq. (34) using Leibnitz's rule

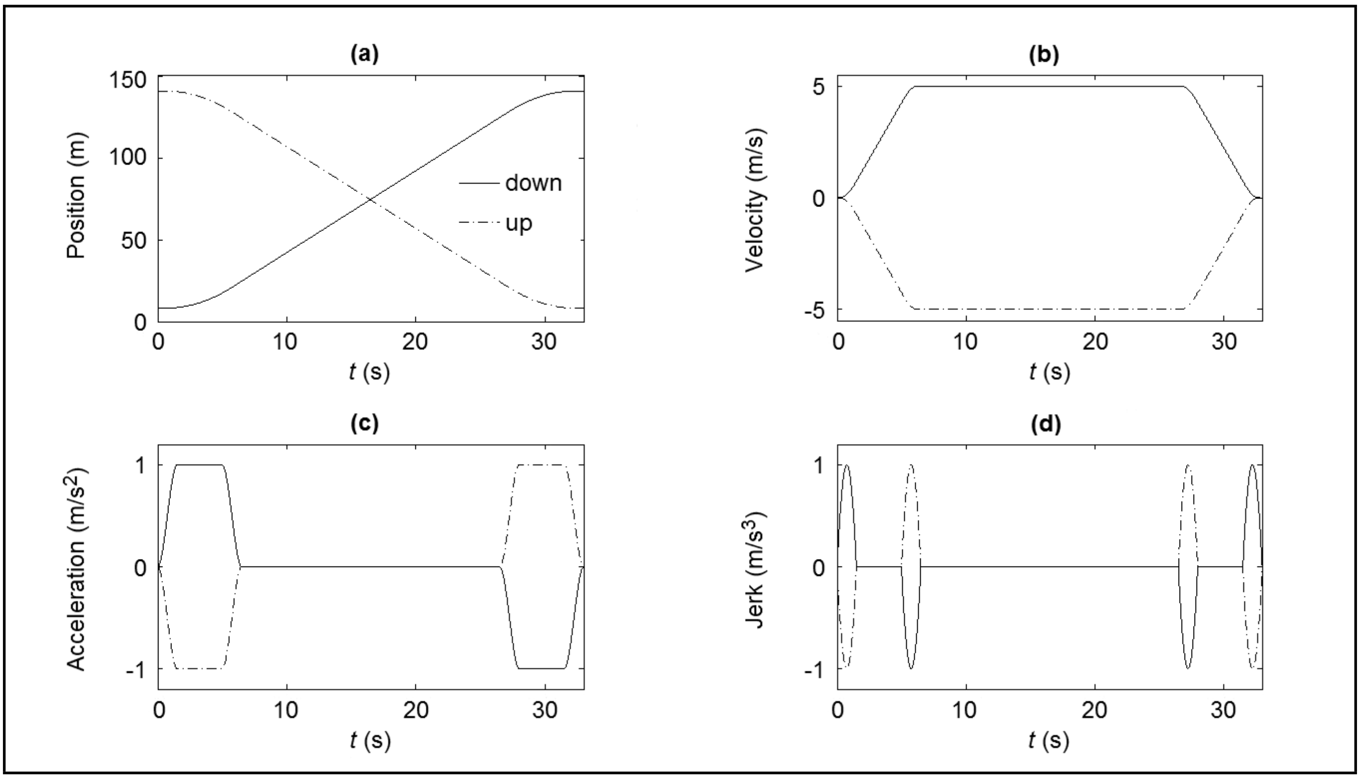

Figure 3. Movement profile of the elevator: (a) $l(t)$; (b) $v(t)$; (c) $\dot{v}(t)$; and (d) $\ddot{v}(t)$.

yields

$$\frac{dT(t)}{dt} = my_t(l(t),t)y_{tt}(l(t),t) + \frac{1}{2}v\left[\rho(y_t+vy_x)^2+\left(Py_x^2+\frac{1}{4}EAy_x^4\right)\right]_{l(t)} + \int_0^{l(t)}\left[\rho(y_t+vy_x)(y_{tt}+\dot{v}y_x+vy_{xt})+\frac{1}{2}P_ty_x^2+Py_xy_{xt}+EAy_x^3y_{xt}\right]dx. \tag{34}$$

Substituting Eq. (16) into Eq. (34), followed by integration by parts, yields

$$\frac{dT(t)}{dt} = -\frac{\rho v^3}{2}\left(y_x^2\right)_0^{l(t)} - \frac{v}{2}\left(Py_x^2\right)_0 + \frac{1}{2}\int_0^{l(t)}\left(P_t+vP_x\right)y_x^2\,dx + \frac{vEA}{2}\left(w_x^4\right)_{l(t)} - \frac{3vEA}{8}\left(w_x^4\right)_0. \tag{35}$$

5. NUMERICAL SIMULATION AND DISCUSSIONS

A typical application of a flexible hoisting string with time-varying length is a traction elevator. The site observation revealed that traction ropes will fiercely vibrate during movement of the elevator. In what follows, the motions of elevator hoisting system were illustrated to evaluate the proposed mathematical models. An elevator hoisting system is modeled as an axially translating string with a rigid body attached at its lower end. In this paper, the flexible hoisting system of a typical high-speed elevator is considered as an example and analyzed.

Table 1. Simulation parameters.

Items	Data values
Density per unit length ρ (kg/m)	0.707
Young's modulus E (Pa)	2.01×10^{11}
String diameter d (m)	14×10^{-3}
Hoisting mass m (kg)	300
Minimum length of the string l_{min} (m)	5
Maximum length of the string l_{max} (m)	140
Maximum velocity v_{max} (m/s)	5
Maximum acceleration a_{max} (m/s^2)	1
Maximum jerk j_{max} (m/s^3)	1
Total travel time t (s)	33
Number of transverse modes n	4

The simulation parameters for the elevator are given in Table 1. The flight time for a travel distance of 135 m (45 stories) is 33 seconds. Figure 3 gives the prescribed displacement, velocity, acceleration and jerk curves of an elevator hoisting system. Utilizing the curves as the input of Eqs. (20), (32), and (35) with the aid of MATLAB® may obtain dynamic responses of an elevator hoisting system. In this work, all numerical analyses were implemented with the aid of MATLAB®.

Consider the free vibration caused by a distributed initial displacement and released from rest. The initial displacement and velocity are respectively

$$y(x,0) = y_0 \sin\frac{x\pi}{l_0}, \qquad y_t(x,0) = 0; \tag{36}$$

where $y_0 = 0.005$ m is the initial amplitude. Transverse vibration responses of the flexible hoisting string at 3 m above the car during movement of the elevator are illustrated in Fig. 4.

Figures 4(a) and 4(b) display reducing vibration amplitudes with an increasing length of the string during downward movement. This is due to the energy of the flexible hoisting sys-

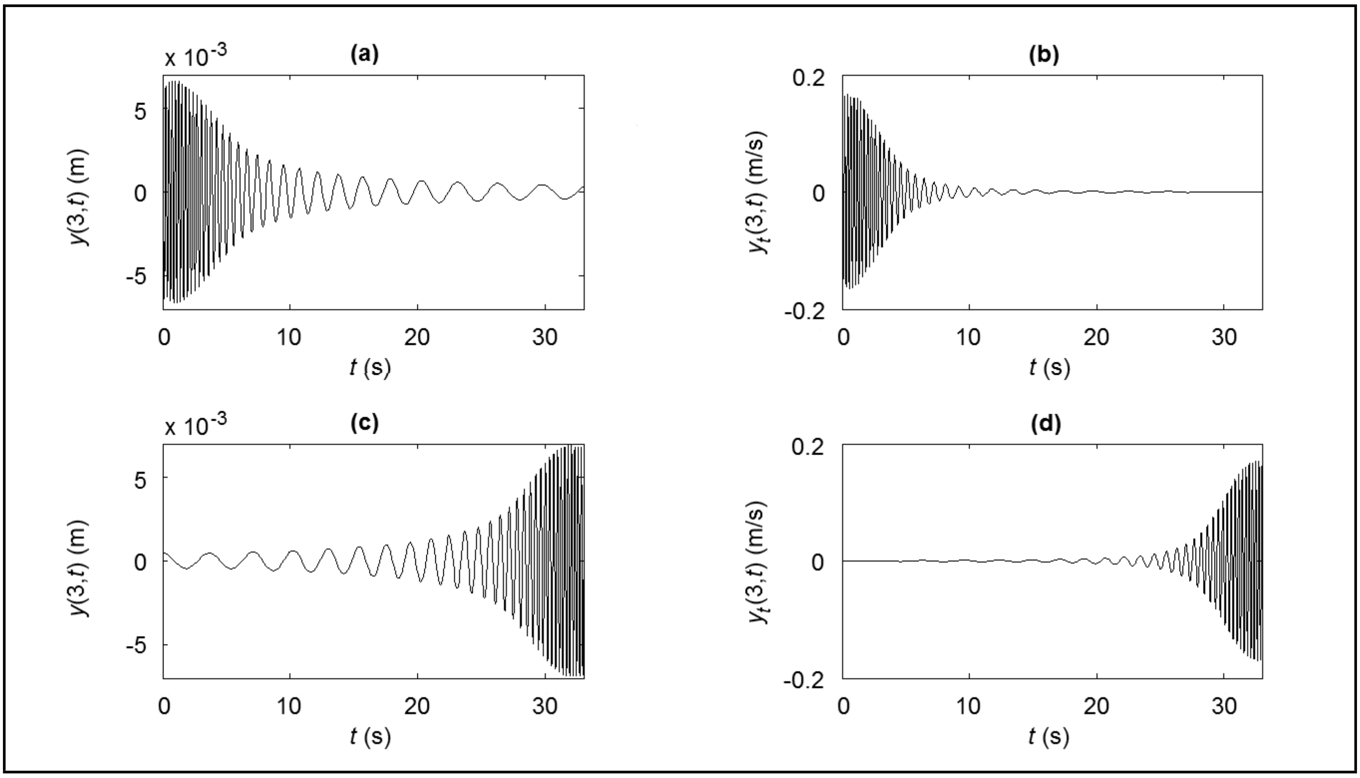

Figure 4. Transverse vibration responses of the flexible hoisting string at 3 m above the car: (a) Transverse vibration displacement during downward movement of the elevator; (b) Transverse vibration velocity during downward movement of the elevator; (c) Transverse vibration displacement during upward movement of the elevator; and (d) Transverse vibration velocity during upward movement of the elevator.

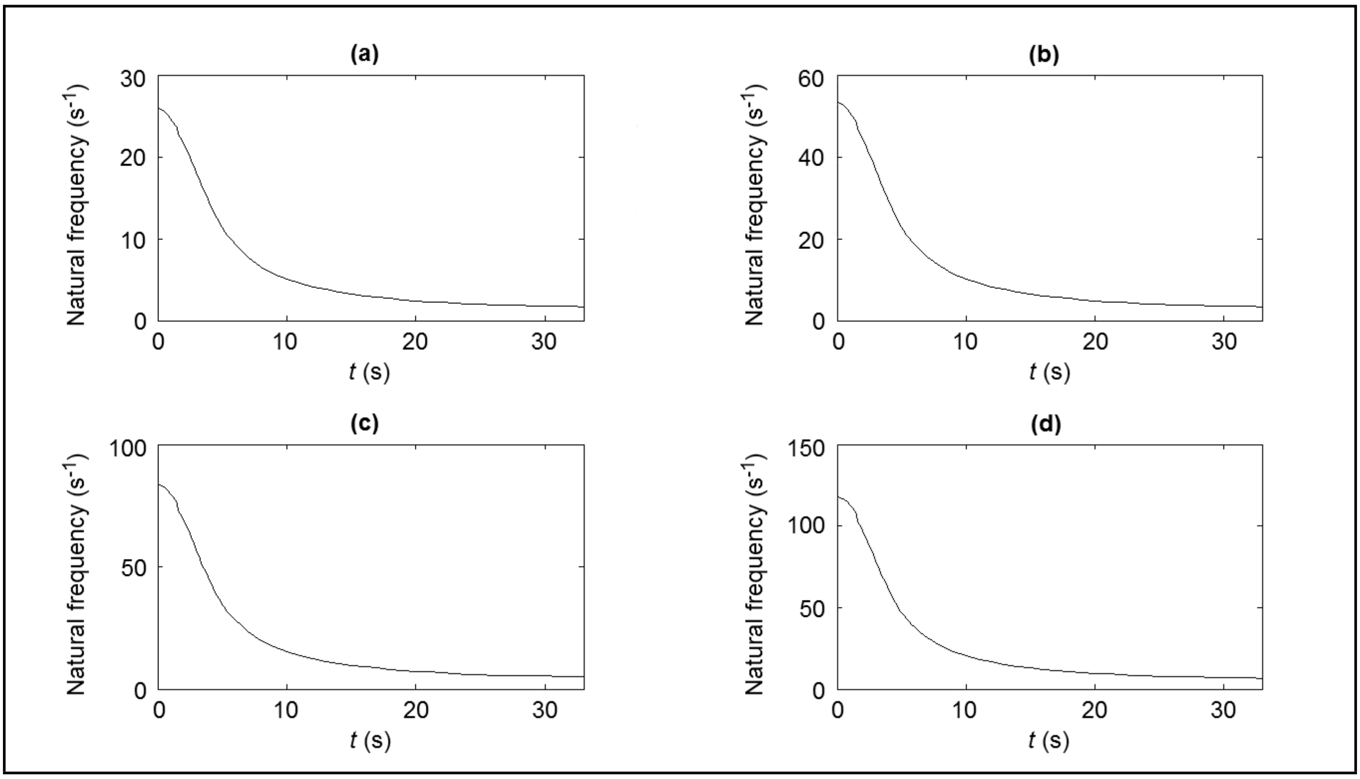

Figure 5. The lowest four order natural frequencies of the flexible hoisting string during downward movement of the elevator: (a) First order natural frequency; (b) Second order natural frequency; (c) Third order natural frequency; and (d) Fourth order natural frequency.

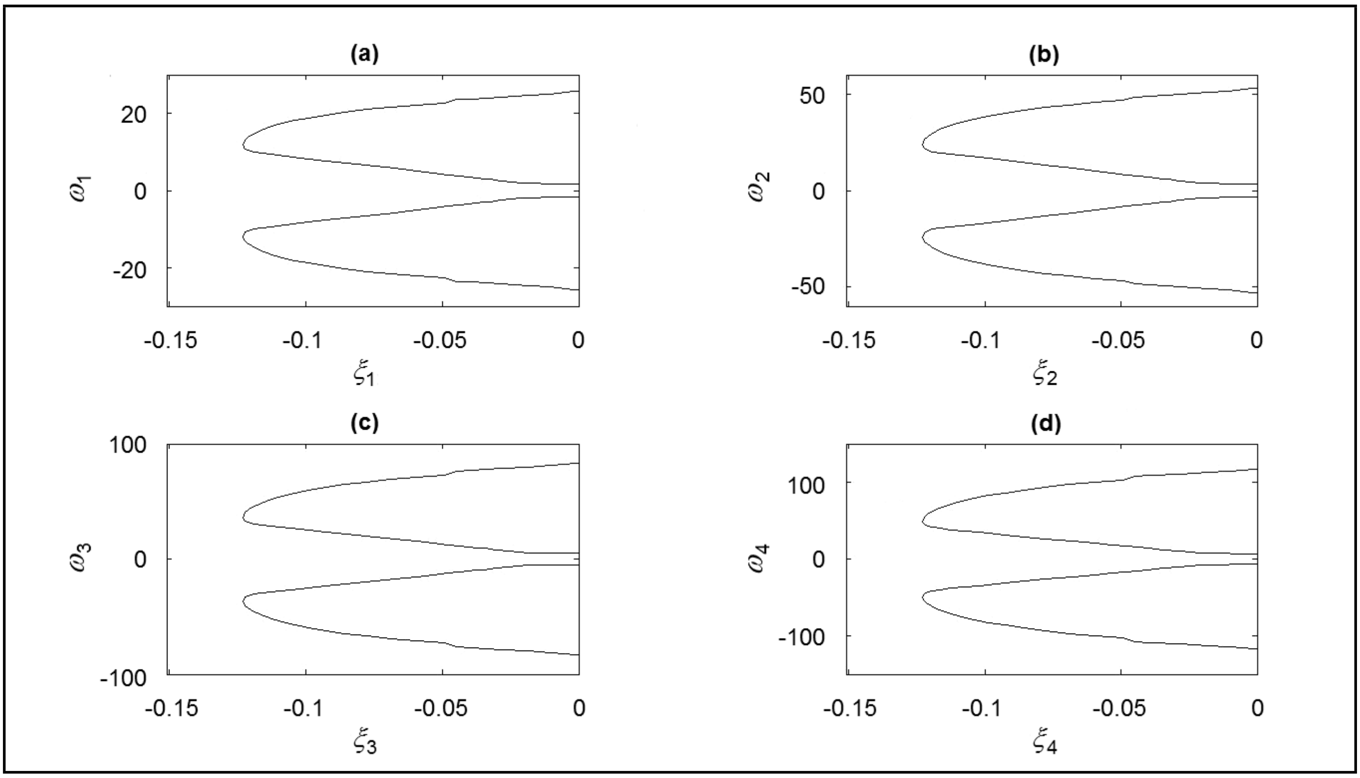

Figure 6. Eigenvalues of the flexible hoisting string during downward movement of the elevator: (a) λ_1; (b) λ_2; (c) λ_3; and (d) λ_4.

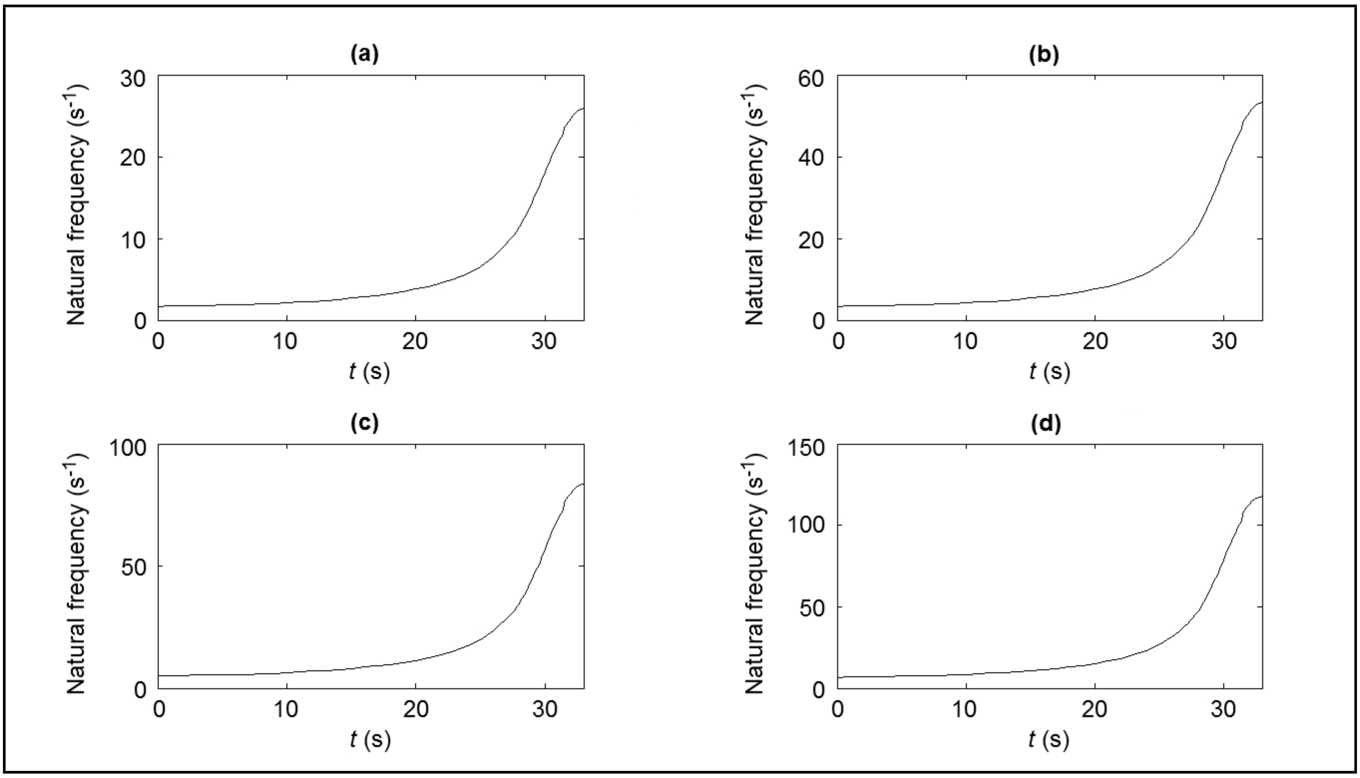

Figure 7. The lowest four order natural frequencies of the flexible hoisting string during upward movement of the elevator: (a) First order natural frequency; (b) Second order natural frequency; (c) Third order natural frequency; and (d) Fourth order natural frequency.

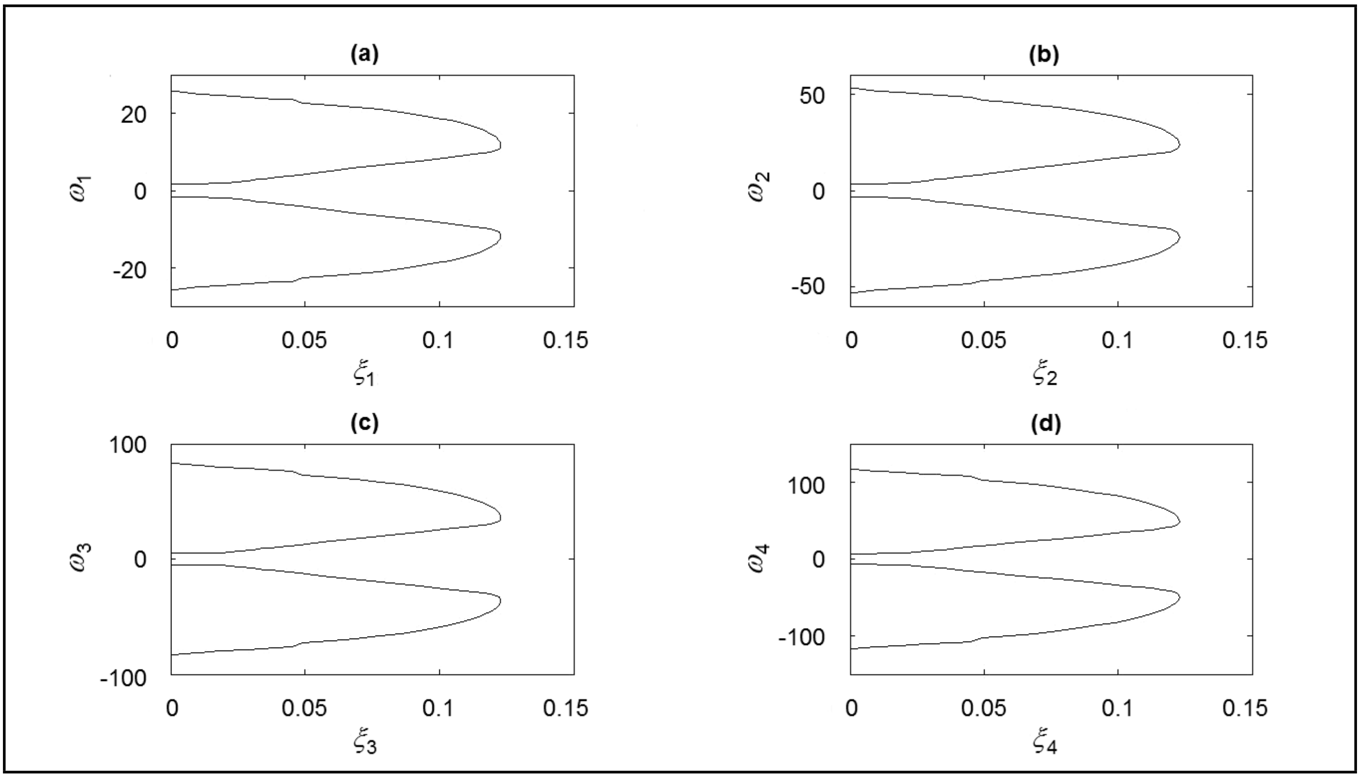

Figure 8. Eigenvalues of the flexible hoisting string during upward movement of the elevator: (a) λ_1; (b) λ_2; (c) λ_3; and (d) λ_4.

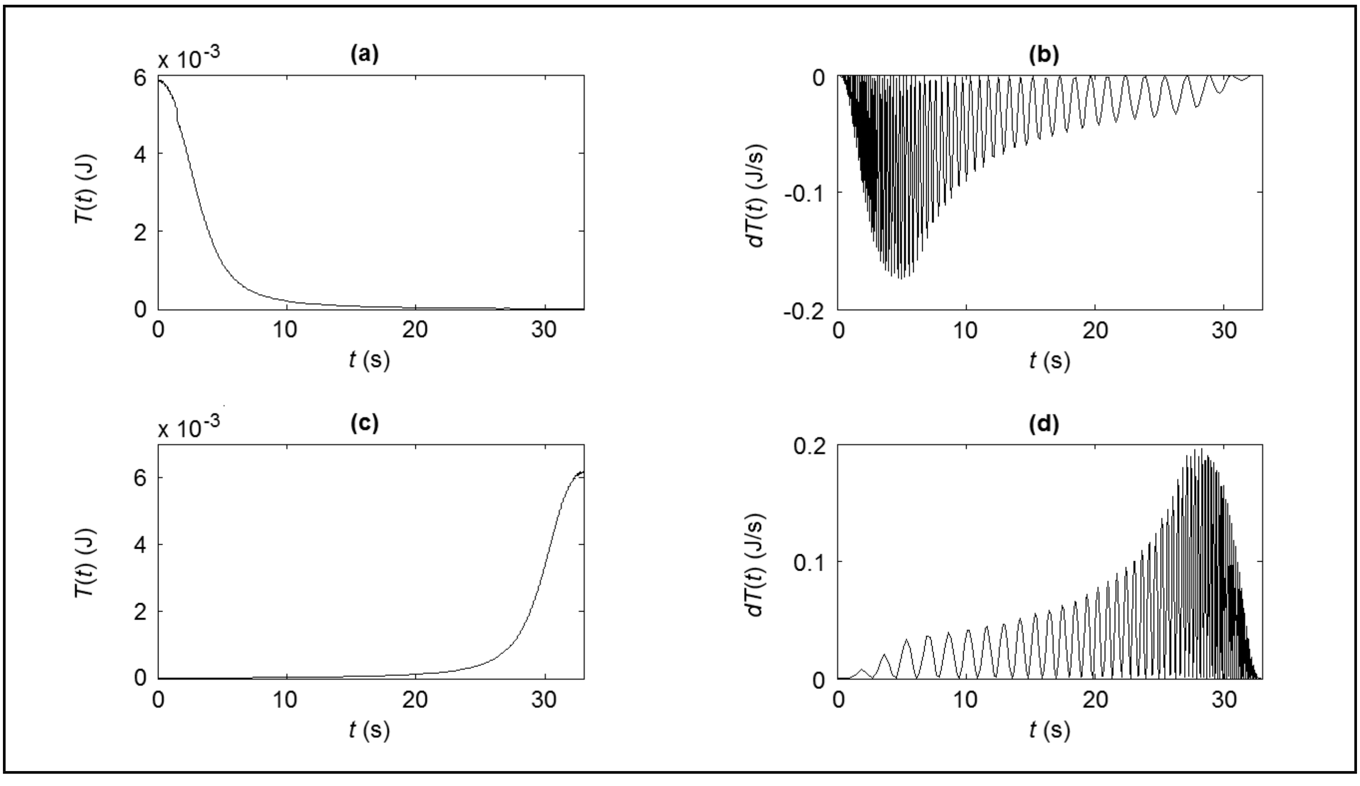

Figure 9. The transverse vibration energy and the rate of change of the energy: (a) The transverse vibration energy during downward movement; (b) The rate of change of the energy during downward movement; (c) The transverse vibration energy during upward movement; and (d) The rate of change of the energy during upward movement.

tem transfers from the transverse vibration to the axial motion by bringing some mass into the domain of effective length, which means that the axially moving string is dissipative during downward movement. A possible physical interpretation of the results is as follows: during downward movement negative external work is required to maintain the prescribed axial motion which, in turn, brings about a convection of mass in the domain of effective length. By contrast, in Fig. 4(c) and Fig. 4(d), we observe that vibration amplitudes of the string increase with the decreasing length of the string during upward movement. This is due to the energy of flexible hoisting system transfers from the axial motion to the transverse vibration by leaving some mass out of the domain of effective length, which means that the axially moving string gains energy during upward movement. A possible physical interpretation of the results is as follows: during upward movement positive external work is required to maintain the prescribed axial motion which, in turn, brings about a convection of mass out of the domain of effective length.

For the stabilization analysis of the flexible hoisting string, the eigenvalues of the system obtained from Eq. (32) should be studied for further consideration. During downward movement of the flexible hoisting system, the mass of the string is increasing and the stiffness of the string is reducing, or the string becomes somewhat stiffer, and the natural frequencies will decrease over time, which has been displayed in Fig. 5. In the mean time, the dynamic motion of the string is stable as the eigenvalues of this system have negative real parts ξ; (see Fig. 6). On the other hand, during upward movement of the flexible hoisting system, the mass of the string is decreasing and the stiffness of the string is increasing, or the string becomes somewhat stiffer, and the natural frequencies will increase with time, (see Fig. 7). At the same time, the dynamic motion of the string is unstable since the real parts ξ of eigenvalues of this system are all positive, which has been shown in Fig. 8. That is, the coupling effect of the translation and transverse motions generates a stabilizing response during downward movement of the flexible hoisting system and a destabilization response during upward movement of the flexible hoisting string. The same conclusions can be reached from the investigation based on an energy standpoint.

The resulting total transverse vibration energy and the rate of change of the energy of the flexible hoisting system are displayed in Fig. 9. During downward movement of the flexible hoisting system, the energy associated with the transverse vibration of the flexible hoisting system decreases, which has been displayed in Fig. 9(a). The reduction of transverse vibration energy translates into the increase in kinetic energy of the flexible hoisting system. In the mean time, the rate of change of the energy $dT(t)/dt$ is negative (see Fig. 9(b)), thus leading to a stabilized transverse dynamic response. By contrast, during upward movement of the flexible hoisting system, the transverse vibration energy of the flexible hoisting system increases, which has been shown in Fig. 9(c). The increase of transverse vibration energy comes from the reduction in kinetic energy of the flexible hoisting system. At the same time, the rate of change of the energy $dT(t)/dt$ is positive (see Fig. 9(d)), therefore resulting in an unstabilized transverse dynamic response. The results explain an inherent unstable shortening cable behavior encountered in the elevator industry.

6. CONCLUSIONS

The linear vibration characteristics and stability for a flexible hoisting string with time-varying length and axial velocity considering coupling of axial movement and flexural deformation were analyzed in this paper. The flexible hoisting system was modeled as an axially moving string with time-varying length and a rigid body at its lower end. The governing equations were derived by using Leibnitz's rule and the extended Hamilton's principle. The Galerkin's method was used to truncate the infinite-dimensional partial differential equations into a set of nonlinear finite-dimensional ordinary differential equations with time-variant coefficients. Based on the numerical simulations, the following conclusions can be obtained:

1. Two different methods, Lyapunov's first method and Lyapunov's second method were used to analyze the stability of the flexible hoisting string with time-varying length and axial velocity. The same results were obtained by the two mehtods.

2. The flexible hoisting string with time-varying length and axial velocity experiences instability during upward movement; the natural frequencies increase because of the reducing mass and the increasing stiffness of the string; and the energy transforms from the axial movement into the flexible deformation. By contrast, it is stable during downward movement; the natural frequencies decrease because of the increasing mass and the reducing stiffness of the string; and the energy coverts from the flexible deformation into the axial movement.

3. The proposed theoretical model and analyses about the stability of the flexible hoisting system in this paper will be helpful for the researchers to comprehend its dynamic behavior and develop the proper method to suppress the vibration in practice.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China (Grant No: 51205249). The authors are also most grateful to the anonymous reviewers and the Editor for their constructive comments.

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