Vibrations of a Rotationally Restrained Circular Plate Resting on a Concentric Rigid Ring Support

Lokavarapu Bhaskara Rao

School of Mechanical & Building Sciences, VIT University, Chennai Campus, Vandalur-Kelambakkam Road, Chennai-600127, Tamil Nadu, India

Chellapilla Kameswara Rao

Department of Mechanical Engineering, Guru Nanak Institutions Technical Campus, Ibrahimpatnam, Hyderabad -501506, A.P, India

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In this paper, the vibrations of a circular plate with a rotationally restrained edge that has concentric rigid ring support are studied. The influences of the rotational restraint parameter and radius of internal rigid ring support on the vibration of the plate's natural frequencies are investigated. Frequencies for the first three modes of vibration are obtained and plotted graphically. The cross-over radius and the optimum location point of internal rigid ring support are determined. The results presented in this paper are from exact analysis, and hence can serve as standard values for estimating the accuracy of results obtained from various approximate methods.

1. INTRODUCTION

In many branches of engineering—such as naval, automobile, and civil—continuous plates are extensively used. There exists a great deal of literature on the present subject area of circular plate vibrations, predominantly with free, clamped, and simply supported edges.[1–4] Leissa[5–10] has reported natural frequency results in many of his papers on continuous circular plates, and other researchers have reported results on the influence of internal rigid ring supports on the dynamic characteristics of circular plates.

Bodine[11] has premeditated the axisymmetric free vibrations of the circular plates, and Laura, et al.[12] presented useful results on the natural frequencies of axisymmetric modes of vibration. The case of the influence of rigid supports along with mode switching was studied by Bodine,[13] varying the values of the ring support radius and Poisson's ratio of the circular plate material. Ding Zhou[14] studied the free vibration of arbitrarily shaped plates with concentric ring elastic and/or rigid supports. In realistic circumstances, to fortify the load-carrying capability of the plate, occasionally internal stiffeners and/or concentric supports are used. In such cases, vibrational characteristics of the plate will vary significantly. Hence, the stiffeners' properties and concentric support should be included in the analysis of the circular plates. Many researchers studied the vibration characteristics of the circular plates with a range of boundary situation and internal intensification.[15–26] Wang[26] studied the problem of fundamental frequency of a circular plate on a ring and free boundary and presented the results for the fundamental frequency related to an anti-symmetric mode of vibration when the support radius is small. Many researchers studied the problem of vibrations of circular plates with concentric ring support, as well.[27–30]

Najafizadeh and Mirkhalaf Valashani[31] carried out the vibration analysis of circular plates that have an eccentric circular perforation and a free edge with an attached concentrated mass at any arbitrary position on the plate. The Rayleigh-Ritz variational method was applied to determine the fundamental natural frequency coefficient for the circular plates with the eccentric circular perforation and arbitrarily attached concentrated mass based on the classical plate theory (CPT). Mirkhalaf Valashani[32] utilized the Rayleigh-Ritz method to investigate the transverse vibration of clamped and simply supported circular plates with an eccentric circular perforation and attached concentrated mass. Wang[33] studied the vibration of a circular plate with an attached core and clamped, simply supported, free and sliding boundary conditions.

However, as we know, in practical industrial engineering situations, we seldom come across such ideal edge conditions. The review of research on the vibration of circular plates restrained against rotation can be found in the studies made by Laura, et al.,[34] Laura and Grossi,[35] Narita and Leissa,[36] Irie, et al.,[37] and Veera, et al.[38] It is well-established that the stipulation on an edge frequently tends to be in between the classical edge conditions (simply supported, free and clamped) and may be in contact with elastic restraints, such as rotational restraints.[39–42] However, there is no other research in the literature addressing the common boundary conditions with a rotational restrained edge at the plate's periphery.

In many practical situations such as bolted connections, the plate edge becomes something between a classical simply supported edge and a clamped edge. Often, the edge conditions can be simulated by using a rotational spring. This is exactly what is attempted in this paper. The main intention of this paper is therefore to study the effect of a rigid ring support radius along a concentric circle, and a plate with a rotationally restrained edge (shown in Fig. 1) using an exact method of solution approach. The natural frequencies of a circular plate for varying values of rotational restraint along the plate edge, and the ring support radius for a wide range of non-dimensional parameters, are presented in graphical form for use in design.

2. ANALYTICAL FORMULATION

Consider a plate of radius R, Poisson's ratio ν, density ρ, modulus of elasticity E, and thickness h. Figure 1 shows a plate which has an outer boundary rotationally restrained and simply supported (radius R), and a rigid ring support at radius bR.

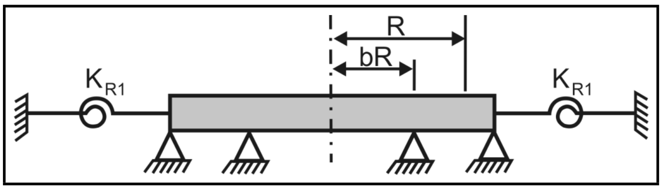

Figure 1. Rotationally restrained circular plate resting on a concentric rigid sing support.

Subscript I denotes $b \leq r \leq 1$ (outer region) and subscript II denotes $0 \leq r \leq b$ (inner region). b, 1 denotes the radius of the inner and outer regions, respectively, after normalizing every length by R. The following fourth-order differential equation[3] describes vibration of plate:

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0. \tag{1}$$

Here, D represents the flexural rigidity of plate. The general form of the lateral displacement of the vibration of a plate can be expressed as $w = u(r)\cos(n\theta)e^{i\omega t}$, where (r, θ) are polar coordinates, w is the transverse displacement, n is the number of nodal diameters, ω is the frequency, and t is time. The function $u(r)$ is a linear combination of Bessel functions $J_n(kr)$, $Y_n(kr)$, $I_n(kr)$, $K_n(kr)$, and $k = R(\rho\omega^2/D)^{1/4}$ is the square root of the non-dimensional frequency.[3] The general solutions for regions I and II are

$$u_{\mathrm{I}}(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr); \tag{2}$$

$$u_{\mathrm{II}}(r) = C_5 J_n(kr) + C_6 I_n(kr). \tag{3}$$

Considering the rotationally restrained and simply supported edge at the outer region, the boundary conditions can be formulated as

$$M_r(r,\theta) = K_{R1}\frac{\partial w_{\mathrm{I}}(r,\theta)}{\partial r}; \tag{4}$$

$$w_{\mathrm{I}}(r,\theta) = 0. \tag{5}$$

The radial moment at the external periphery is expressed as

$$M_r(r,\theta) = -\frac{D}{R}\left[\frac{\partial^2 w_{\mathrm{I}}(r,\theta)}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w_{\mathrm{I}}(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_{\mathrm{I}}(r,\theta)}{\partial \theta^2}\right)\right]. \tag{6}$$

From Eqs. (4) and (6) yields the following expression:

$$\left[\frac{\partial^2 w_{\mathrm{I}}(r,\theta)}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w_{\mathrm{I}}(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_{\mathrm{I}}(r,\theta)}{\partial \theta^2}\right)\right] = -R_{11}\frac{\partial w_{\mathrm{I}}(r,\theta)}{\partial r}. \tag{7}$$

Equations (5) and (7) yield the following:

$$u_{\mathrm{I}}''(r) + \nu\left[u_{\mathrm{I}}'(r) - n^2 u_{\mathrm{I}}(r)\right] = -R_{11}u_{\mathrm{I}}'(r); \tag{8}$$

$$u_{\mathrm{I}}(r) = 0. \tag{9}$$

At the outer region (at $r = 1$), the boundary conditions are as follows:

$$u_{\mathrm{I}}''(1) + \nu\left[u_{\mathrm{I}}'(1) - n^2 u_{\mathrm{I}}(1)\right] = -R_{11}u_{\mathrm{I}}'(1); \tag{10}$$

$$u_{\mathrm{I}}(1) = 0; \tag{11}$$

where $R_{11} = \frac{K_{R1}R}{D}$ is the normalized spring constant K_{R1} of the rotational elastic spring at outer periphery.

Apart from the rotationally restrained boundary at the outer edge, the continuity requirements at concentric ring ($r = b$) are as follows:

$$u_{\mathrm{I}}(b) = 0; \tag{12}$$

$$u_{\mathrm{II}}(b) = 0; \tag{13}$$

$$u_{\mathrm{I}}'(b) = u_{\mathrm{II}}'(b); \tag{14}$$

$$u_{\mathrm{I}}''(b) = u_{\mathrm{II}}''(b). \tag{15}$$

The non-trivial solutions to Eqs. (10)–(15) are required. From Eqs. (2), (3), and (10)–(15), we obtained the subsequent equations:

$$\begin{aligned}
&\left[\frac{k^2}{4}P_2 + \frac{k}{2}(\nu + R_{11})P_1 - \left(\frac{k^2}{2} + \nu n^2\right)J_n(k)\right]C_1 + \\
&\left[\frac{k^2}{4}Q_2 + \frac{k}{2}(\nu + R_{11})Q_1 - \left(\frac{k^2}{2} + \nu n^2\right)Y_n(k)\right]C_2 + \\
&\left[\frac{k^2}{4}R_2 + \frac{k}{2}(\nu + R_{11})R_1 + \left(\frac{k^2}{2} - \nu n^2\right)I_n(k)\right]C_3 - \\
&\left[\frac{k^2}{4}S_2 - \frac{k}{2}(\nu + R_{11})S_1 + \left(\frac{k^2}{2} - \nu n^2\right)K_n(k)\right]C_4 = 0;
\end{aligned} \tag{16}$$

$$[J_n(k)]C_1 + [Y_n(k)]C_2 + [I_n(k)]C_3 + [K_n(k)]C_4 = 0; \tag{17}$$

$$J_n(kb)C_1 + Y_n(kb)C_2 + I_n(kb)C_3 + K_n(kb)C_4 = 0; \tag{18}$$

$$J_n(kb)C_5 + I_n(kb)C_6 = 0; \tag{19}$$

$$\begin{aligned}
&\left[\frac{k}{2}P_1'\right]C_1 + \left[\frac{k}{2}Q_1'\right]C_2 + \left[\frac{k}{2}R_1'\right]C_3 - \left[\frac{k}{2}S_1'\right]C_4 - \\
&\left[\frac{k}{2}P_1'\right]C_5 - \left[\frac{k}{2}R_1'\right]C_6 = 0;
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\left[\frac{k^2}{4}P_2' - \frac{k^2}{2}J_n(kb)\right]C_1 + \left[\frac{k^2}{4}Q_2' - \frac{k^2}{2}Y_n(kb)\right]C_2 + \\
&\left[\frac{k^2}{4}R_2' + \frac{k^2}{2}I_n(kb)\right]C_3 + \left[\frac{k^2}{4}S_2' + \frac{k^2}{2}K_n(kb)\right]C_4 - \\
&\left[\frac{k^2}{4}P_2' - \frac{k^2}{2}J_n(kb)\right]C_5 - \left[\frac{k^2}{4}R_2' + \frac{k^2}{2}I_n(kb)\right]C_6 = 0;
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
P_1 &= J_{n-1}(k) - J_{n+1}(k); & P_2 &= J_{n-2}(k) + J_{n+2}(k); \\
Q_1 &= Y_{n-1}(k) - Y_{n+1}(k); & Q_2 &= Y_{n-2}(k) + Y_{n+2}(k); \\
R_1 &= I_{n-1}(k) + I_{n+1}(k); & R_2 &= I_{n-2}(k) + I_{n+2}(k); \\
S_1 &= K_{n-1}(k) + K_{n+1}(k); & S_2 &= K_{n-2}(k) + K_{n+2}(k); \\
P_1' &= J_{n-1}(kb) - J_{n+1}(kb); & P_2' &= J_{n-2}(kb) + J_{n+2}(kb); \\
Q_1' &= Y_{n-1}(kb) - Y_{n+1}(kb); & Q_2' &= Y_{n-2}(kb) + Y_{n+2}(kb); \\
R_1' &= I_{n-1}(kb) + I_{n+1}(kb); & R_2' &= I_{n-2}(kb) + I_{n+2}(kb); \\
S_1' &= K_{n-1}(kb) + K_{n+1}(kb); & S_2' &= K_{n-2}(kb) + K_{n+2}(kb).
\end{aligned}$$

3. SOLUTION

For the given values of n, ν, R_{11}, and b Eqs. (16)–(21) derived above are solved to obtain an exact characteristic frequency equation by suitably eliminating the coefficients C_1, C_2, C_3, C_4, C_5, and C_6. The frequency parameter k can be

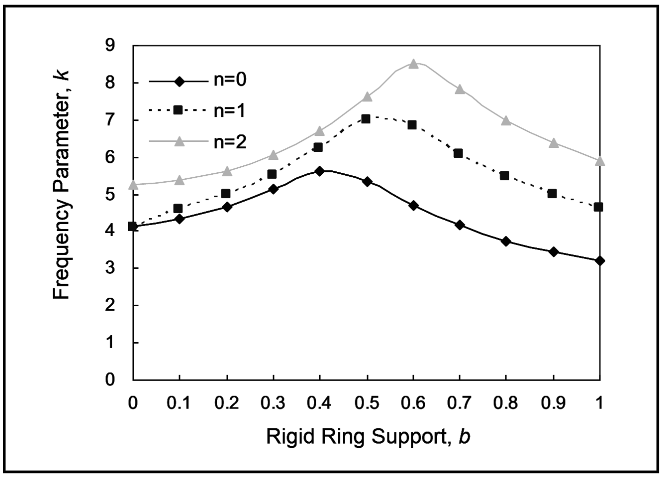

Figure 2. Frequency of a circular plate and concentric rigid ring support radius b for $R_{11} = 2.5$.

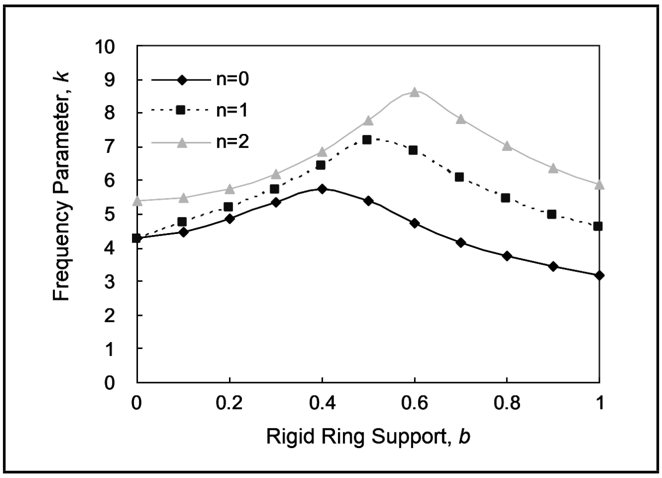

Figure 3. Fundamental frequency k of a circular plate and concentric rigid ring support radius b for $R_{11} = 5$.

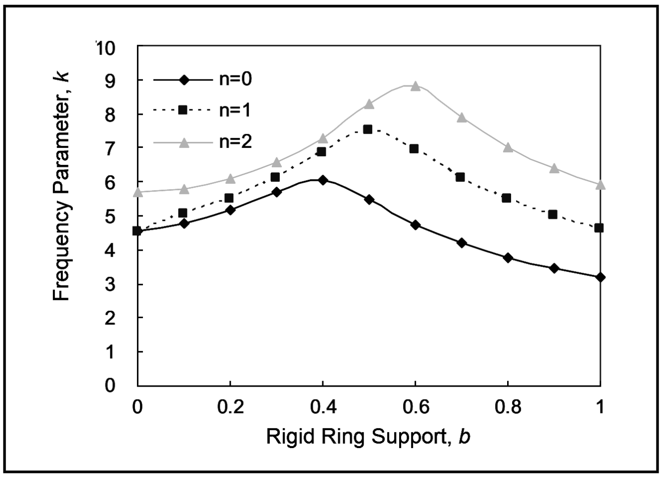

Figure 4. Fundamental frequency k of a circular plate and concentric rigid ring support radius b for $R_{11} = 20$.

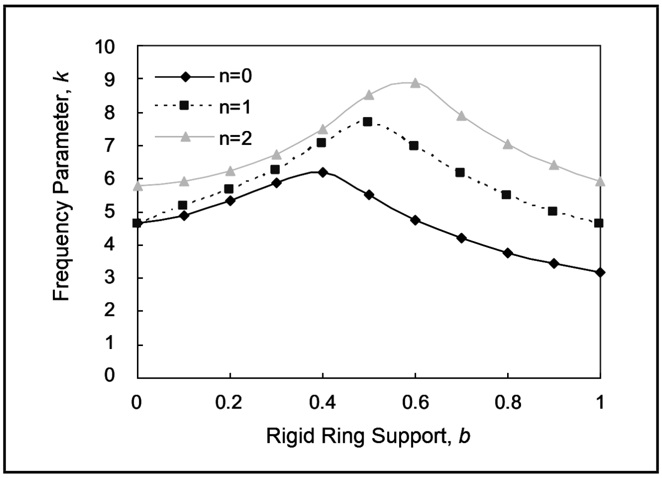

Figure 5. Fundamental frequency k of a circular plate and concentric rigid ring support radius b for $R_{11} = 50$.

determined from the characteristic equation by a simple root search method. Using Mathematica, computer software with symbolic capabilities is used to solve this problem. Poisson's ratio utilized in these studies is 0.3.

4. RESULTS AND DISCUSSION

The fundamental frequency parameters for the first three modes for diverse values of rotational restraints ($R_{11} = 2.5$, 5, 20, 50, 100, 500, 1000, and 10^{16}) are computed. Results for first three modes of vibrations are determined and presented in Figs. 2–9. As seen in Fig. 2, for a particular value of $R_{11} = 2.5$, the curve is unruffled with different segments because of mode switching. The fundamental frequency is related to asymmetric $n = 1$ mode when the concentric ring support radius b is small. Within this segment (the spotted lines in Fig. 2), the fundamental frequency reduces as the value of ring support radius b decreases. On the higher concentric rigid support radius, the fundamental frequency is related to the axisymmetric mode. Within the other segment (the continuous lines in Fig. 2), the fundamental frequency increases as b increases up to a peak point, corresponding to the maximum frequency, and thereafter decreases as b increases in value, as shown in Fig. 2.

Mode switching takes place at $b = 0.012427$, and the fundamental frequency parameter depends on the asymmetric mode when $b \leq 0.012427$ (as shown by spotted lines in Fig. 2). When the value of b goes beyond 0.012427, the axisymmetric mode leads to an accurate fundamental frequency, as shown by constant lines in Fig. 2. The optimum location is the critical radial point for the concentric rigid ring support corresponding to the maximum frequency parameter. The optimum location of concentric rigid ring support and corresponding fundamental frequencies are $b = 0.4$ and $k = 5.62304$, respectively, which are equal to the nodal radius related to the axisymmetric mode and its frequency.

Similarly, it has been observed from Figs. 3–9, for varying values of the rotational restraint ($R_{11} = 2.5$, 5, 20, 50, 100, 500, 1000, and 10^{16}) parameter, that the curve is unruffled with different segments because of the switching of vibration modes. The fundamental frequency is related to the asymmetric $n = 1$ mode when the concentric ring support radius b is small. Within this segment (the spotted lines in Figs. 3–9), the fundamental frequency reduces as the value of the ring support radius b decreases. On higher values of the concentric rigid support radius, the fundamental frequency is related to the axisymmetric mode. Within this segment (the continuous lines in Figs. 3–9), the fundamental frequency increases as b decreases up to a peak point corresponding to the maximum frequency, and thereafter decreases as b decreases in value, as shown in Figs. 3–9. The cross-over radius is the radius of the ring support where the switching of vibration mode occurs. The cross-over radius b_{cor} and the corresponding frequency parameters

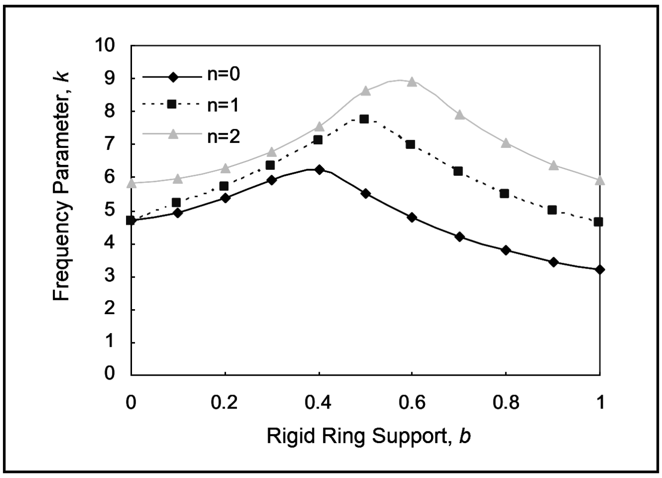

Figure 6. Fundamental frequency k of a circular plate and concentric rigid ring support radius b for $R_{11} = 100$.

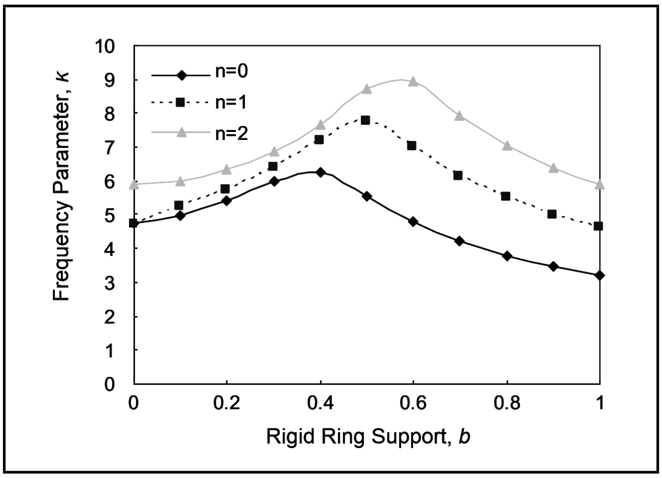

Figure 7. Fundamental frequency k of a circular plate and concentric rigid ring support radius b for $R_{11} = 500$.

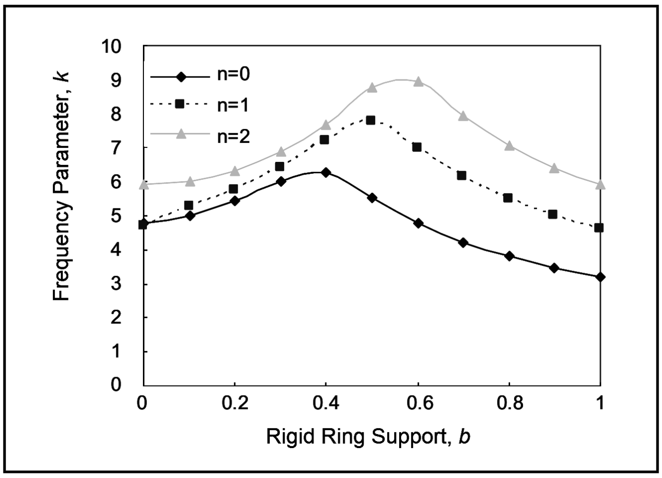

Figure 8. Fundamental frequency k of a circular plate and concentric rigid ring support radius b for $R_{11} = 1000$.

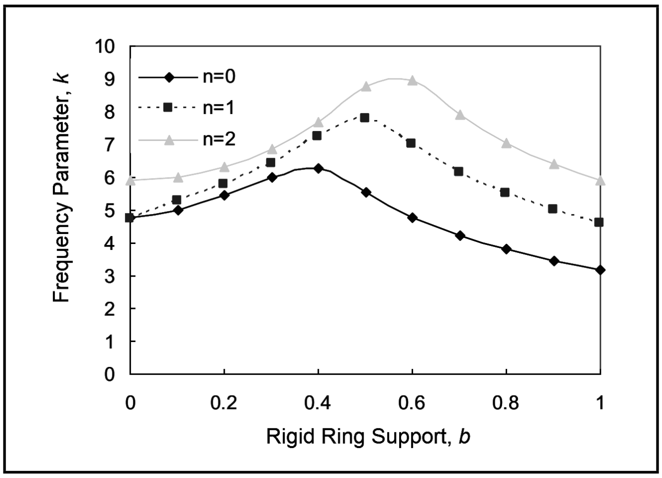

Figure 9. Fundamental frequency k of a circular plate and concentric rigid ring support radius b for $R_{11} = 10^{16}$.

k_{cor} are determined and presented in Table 1. In addition, the optimal solutions (the optimal position of concentric rigid ring support b_{opt} and subsequent fundamental frequency k_{opt}) are determined and presented in Table 2.

From the results obtained, it can be observed that the cross-over radius increases from 0.01242 to 0.017216 as the rotational restraint parameter R_{11} varies from 2.5 to 10^{16}. The optimal location is 0.4, which remains constant from $R_{11} = 2.5$ to 0.4 for $R_{11} = 10^{16}$. However, the fundamental frequency increases from 5.62304 to 6.27065 at the respective optimal locations.

In addition to the variation of the fundamental frequency parameters for the first three modes for diverse values of rotational restraints ($R_{11} = 2.5$, 5, 20, 50, 100, 500, 1000, and 10^{16}) as shown in Figs. 2–9, the percentage of variation of frequency due to the rotational restraints are computed. The percentage increment in the frequency parameter for the first three modes is presented in Table 3. It is noted that for a given radius, the percentage increment decreases for the first three modes ($n = 0$, $n = 1$, and $n = 2$) as the rotational restraint parameter R_{11} varies from 2.5 to 10^{16} in each case, i.e. for the first three modes. The percentage of variation of frequency as the rotational restraint parameter R_{11} varies from 2.5 to 10^{16} for the first three modes is shown in Fig. 10. It is noted from Fig. 10, which, for a given mode and as the rotational restraint parameter R_{11} varies from 2.5 to 10^{16}, that the percentage increment in frequency increases first and then decreases as the rigid ring radius parameter increases. Also, it is observed that there is a cross-over radius of the variation of frequency with three modes.

Results of this type are not available in the published literature. The values of the fundamental frequency for the axisymmetric mode very closely resemble those presented by Laura, et al.[12] A comparison of the results is shown in Table 4, wherein the values of the exact fundamental frequency for the plate with a free boundary (setting $R_{11} \to 0$ to the current predicament) are compared to those presented by Wang.[26] From a realistic point of view, when the rotational stiffness parameter becomes small, the edge tends to become a quasi-simply supported edge, and when it becomes larger, it tends to become very close to that of a clamped edge.

5. CONCLUSIONS

The fundamental frequencies of a rotationally restrained circular plate resting on a concentric rigid ring support have been obtained for a wide range of parametric values in this paper. It can easily be seen that the fundamental mode of frequency switches from $n = 1$ to $n = 0$ at a specific radius of the concentric ring. Mode switching is noted and computed exactly. The optimal solutions for internal concentric rigid ring support and the corresponding fundamental frequency are computed exactly, and the results are obtained from closed form solu-

Table 1. The cross-over radius, b_{cor} and the corresponding frequency parameters, k_{cor}.

R_{11}	2.5	5	20	50	100	500	1000	10^{16}
b_{cor}	0.01242	0.01436	0.01640	0.01704	0.01738	0.01713	0.01682	0.017216
k_{cor}	4.16626	4.32284	4.60973	4.71843	4.76269	4.7977	4.79848	4.80809

Table 2. Optimal locations (concentric rigid support, b_{opt} and subsequent frequency, k_{opt}).

R_{11}	2.5	5	20	50	100	500	1000	10^{16}
b_{opt}	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
k_{opt}	5.62304	5.77268	6.06861	6.17972	6.22338	6.26089	6.26567	6.27065

Table 3. Percentage increment of frequency for the first there modes as the rotational restraint parameter R_{11} varies from 2.5 to 10^{16}.

b	$n=0$ ($R_{11}=2.5$ to 10^{16})	$n=1$ ($R_{11}=2.5$ to 10^{16})	$n=2$ ($R_{11}=2.5$ to 10^{16})
0	15.04128004	14.74715067	11.87518234
0.1	15.5284128	14.75920692	12.14681841
0.2	16.1720786	15.21135106	12.82798834
0.3	16.35934421	15.80606952	13.74803696
0.4	11.51707973	16.13928891	14.74936342
0.5	3.341207227	11.22475509	15.10117797
0.6	1.928786856	2.289155392	5.179809962
0.7	1.531773363	1.262661622	1.178654743
0.8	1.188432836	0.988562572	0.83312529
0.9	0.706107481	0.639596197	0.580342714
1	0	0	0

Table 4. Comparison of fundamental frequency for $\nu = 0.3$, with Wang,[26] for free edge.

Ring support radius, b	Wang[26]	Present
0	0	0
0.02	1.501	1.50077
0.05	1.634	1.63422
0.1	1.789	1.78911
0.15	1.922	1.92226
0.2	2.051	2.05103

tions. Thus, the results presented in this paper are expected to serve as benchmark solutions for comparison to those from approximated methods. The exact results presented in the various graphs and tables included in this paper are also expected to be of use in various design-engineering applications.

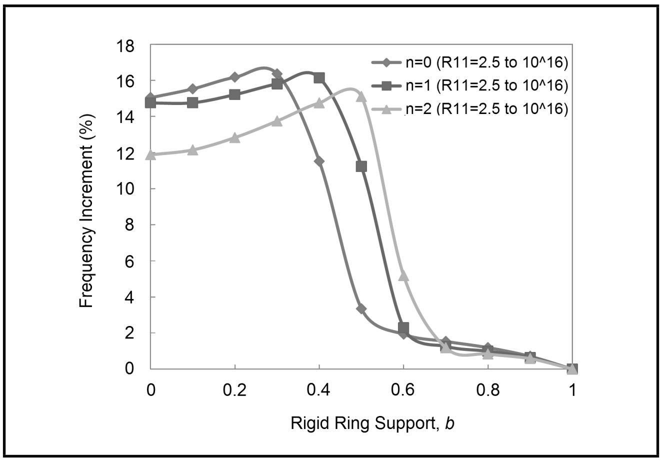

Figure 10. The percentage of variation of frequency as the rotational restraint parameter R_{11} varies from 2.5 to 10^{16} for the first three modes.

REFERENCES

1 Timoshenko, S. and Woinowsky-Krieger, S. *Theory of Plates and Shells*, McGraw-Hill, New York, (1959).

2 Szilard, R. *Theory and Analysis of Plates*, Prentice-Hall, New Jersey, (1974).

3 Leissa, A. W. *Vibration of Plates*, NASA SP-160, (1969).

4 Mcleod, A. J. and Bishop, R. E. D. The forced vibration of circular flat plates, *Mechanical Engineering Science Monograph*, **1**, 1–33, (1965).

5 Leissa, A. W. Recent research in plate vibrations: classical theory, *Shock and Vibration Digest*, **9** (10), 13–24, (1977). DOI:10.1177/058310247700901005

6 Leissa, A. W. Recent research in plate vibrations: complicating effects, *Shock and Vibration Digest*, **10** (12), 21–35, (1977). DOI:10.1177/058310247801001204

7 Leissa, A. W. Plate vibration research, 1976–1980: classical theory, *Shock and Vibration Digest*, **13** (9), 11–22, (1981). DOI:10.1177/058310248101300905

8 Leissa, A. W. Plate vibration research, 1976–1980: complicating effects, *Shock and Vibration Digest*, **13** (10), 17–36, (1981). DOI:10.1177/058310248101301004

9 Leissa, A. W. Recent research in plate vibrations, 1981–1985, Part I. Classical theory, *Shock and Vibration Digest*, **19** (2), 11–18, (1987). DOI:10.1177/058310248701900204

10 Leissa, A. W. Recent research in plate vibrations, 1981–1985, Part II. Complicating effects, *Shock and Vibration Digest*, **19** (3), 10–24, (1987). DOI:10.1177/058310248701900304

11 Bodine, R. Y. The fundamental frequency of a thin, flat circular plate supported along a circle of arbitrary radius, *Journal of Applied Mechanics*, **26**, 666–668, (1959).

12 Laura, P. A. A., Gutierrez, R. H., Vera, S. A. and Vega, D. A. Transverse vibrations of a circular plate with a free edge and a concentric circular support, *Journal of Sound and Vibration*, **223** (5), 842–845, (1999). DOI:10.1006/jsvi.1998.2150

13 Bodine, R. Y. Vibration of Circular Plate supported by a concentric ring of arbitrary radius, *Journal of Acoustical Society of America*, **41**, 1551–1551, (1967).

14 Zhou, Ding Free vibration of arbitrarily shaped plates with concentric ring elastic and/or rigid supports, *Computers & Structures*, **50** (5), 685–692, (1994). DOI:10.1016/0045-7949(94)90427-8

15 Magrab, E. B. *Vibrations of Elastic Structural Members*, Sijthoff & Noordhoff, The Netherlands, (1979).

16 Weisensel, G. N. Natural frequency information for circular and annular plates, *Journal of Sound and Vibration*, **133** (1), 129–134, (1989). DOI:10.1016/0022-460x(89)90987-5

17 Azimi, S. Free vibration of circular plates with elastic or rigid interior support, *Journal of Sound and Vibration*, **120** (1), 37–52, (1988). DOI:10.1016/0022-460x(88)90333-1

[18] Lam, K. Y. and Liew, K. M. A numerical model based on orthogonal plate functions for vibration of ring supported elliptical plates, *Computational Mechanics*, **9** (2), 113–126, (1992). DOI:10.1007/bf00370066

[19] Wang, C. M. and Thevendran, V. Vibration analysis of annular plates with concentric supports using a variant of Rayleigh-Ritz method, *Journal of Sound and Vibration*, **163** (1), 137–149, (1993). DOI:10.1006/jsvi.1993.1153

[20] Liew, K. M. and Lam, K.Y. Transverse vibration of solid circular plates continuous over multiple concentric annular supports, *Transactions of ASME. Journal of Applied Mechanics*, **60** (1), 208–210, (1993). DOI:10.1115/1.2900749

[21] Liew, K. M., Xiang, Y., Wang, C. M. and Kitipornchai, S. Flexural vibration of shear deformable circular and annular plates on ring supports, *Computer Methods in Applied Mechanics and Engineering*, **110** (3–4), 301–315, (1993). DOI:10.1016/0045-7825(93)90211-f

[22] Liew, K. M., Xiang, Y., Kitipornchai, S. and Wang, C. M. Buckling and vibration of annular Mindlin plates with internal concentric ring supports subject to in-plane radial pressure, *Journal of Sound and Vibration*, **177** (5), 689–707, (1994). DOI:10.1006/jsvi.1994.1461

[23] Xiang, Y. Vibration of Circular Mindlin Plates with concentric elastic ring support, *International Journal of Mechanical Sciences*, **45** (3), 497–517, (2003). DOI:10.1016/s0020-7403(03)00059-6

[24] Wang, C. Y. and Wang, C. M. Fundamental Frequencies of Circular Plates with internal elastic ring support, *Journal of Sound and Vibration*, **263** (5), 1071–1078, (2003). DOI:10.1016/s0022-460x(03)00275-x

[25] Bhaskara Rao, L. and Kameswara Rao, C. Buckling of circular plates with an internal elastic ring support and elastically restrained guided edge against translation, *Mechanics Based Design of Structures and Machines*, **37** (1), 60–72, (2009). DOI:10.1080/15397730802706672

[26] Wang, C. Y. On the fundamental frequency of a circular plate supported on a ring, *Journal of Sound and Vibration*, **243** (5), 945–946, (2001). DOI:10.1006/jsvi.2000.3463

[27] Kunukkasseril, V. X. and Swamidas, A. S. J. Vibration of continuous circular plates, *International Journal of Solids and Structures*, **10** (6), 603–619, (1974). DOI:10.1016/0020-7683(74)90045-6

[28] Singh, A. V. and Mirza, S. Free axisymmetric vibration of a circular plate elastically supported along two concentric circles, *Journal of Sound and Vibration*, **48** (3), 425–429, (1976). DOI:10.1016/0022-460x(76)90068-7

[29] Sasaki, M. and Chonan, S. Vibration and stability of elastically supported circular plates under conservative and non-conservative loads, *Journal of Sound and Vibration*, **103** (1), 99–108, (1985). DOI:10.1016/0022-460x(85)90249-4

[30] Azimi, S. Free vibration of circular plates with elastic edge supports using the receptance method, *Journal of Sound and Vibration*, **120** (1), 19–35, (1987). DOI:10.1016/0022-460x(88)90332-x

[31] Najafizadeh, M. M. and Mirkhalaf Valashani, S. M. Vibration analysis of circular plates with an eccentric circular perforation with a free edge and attached concentrated mass at any arbitrary position, *Australian Journal of Basic and Applied Sciences*, **5** (12), 3052–3058, (2011).

[32] Mirkhalaf Valashani, S. M. Transverse vibration of clamped and simply supported circular plates with an eccentric circular perforation and attached concentrated mass, *Journal of Solid Mechanics*, **1**, 37–44, (2009).

[33] Wang, C. Y. Vibration of a circular plate with an attached core, *Journal of Sound and Vibration*, **280** (3–5), 1075–1082, (2005). DOI:10.1016/j.jsv.2004.02.029

[34] Laura, P. A. A., Paloto, J. C., and Santos, R. D. A note on the vibration and stability of a circular plate elastically restrained against rotation, *Journal of Sound and Vibration*, **41** (2), 177–180, (1975). DOI:10.1016/s0022-460x(75)80095-2

[35] Laura, P. A. A. and Grossi, R. O. Influence of Poisson's ratio on the lower natural frequencies of transverse vibration of a circular plate of linearly varying thickness and with an edge elastically restrained against rotation, *Journal of Sound and Vibration*, **60** (4), 587–590, (1978). DOI:10.1016/s0022-460x(78)80094-7

[36] Narita, Y. and Leissa, A. W. Transverse vibration of simply supported circular plates having partial elastic constraints, *Journal of Sound and Vibration*, **70** (1), 103–116, (1980). DOI:10.1016/0022-460x(80)90557-x

[37] Irie, T., Yamada, G., and Tanaka, K. Free vibration of circular plate elastically restrained along some radial segments, *Journal of Sound and Vibration*, **89** (3), 295–308, (1983). DOI:10.1016/0022-460x(83)90538-2

[38] Vera, S. A., Febbo, M., Rossit, C. A., and Dolinko, A. E. Transverse vibrations of circular annular plates with edges elastically restrained against rotation, used in acoustic underwater transducers, *Ocean Engineering*, **29** (10), 1201–1208, (2002). DOI:10.1016/s0029-8018(01)00083-x

[39] Kim, C. S. and Dickinson, S. M. The Flexural Vibration of the Isotropic and Polar Orthotropic Annular and Circular Plates with elastically restrained peripheries, *Journal of Sound and Vibration*, **143** (1), 171–179, (1990). DOI:10.1016/0022-460x(90)90576-l

[40] Wang, C. Y. and Wang, C. M. Buckling of Circular Plates with an internal ring support and elastically restrained edges, *Thin-walled Structures*, **39** (9), 821–825, (2001). DOI:10.1016/s0263-8231(01)00031-3

[41] Bhaskara Rao, L. and Kameswara Rao, C. Buckling of annular plates with elastically restrained external and internal edges, *Mechanics Based Design of Structures and Machines*, **41** (2), 222–235, (2013). DOI:10.1080/15397734.2012.717495

[42] Bambill, D. V., La Malfa, S., Rossit, C. A. and Laura, P. A. A. Analytical and experimental investigation on transverse vibrations of solid, circular and annular plates carrying a concentrated mass at an arbitrary position with marine applications, *Ocean Engineering*, **31** (2), 127–138, (2004). DOI:10.1016/s0029-8018(03)00116-1