Non-linear Thermally Induced Vibrations of Non-homogeneous Rectangular Plate of Linearly Varying Thickness in the Presence of External Force

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An analysis with numerical results is presented for non-linear thermally induced forced vibrations of rectangular plate of variable thickness on the basis of classical plate theory. The thickness of the plate is considered as linearly varying in x-direction. Approximate formulae are proposed for estimating the maximum deflection of a rectangular plate subject to a uniformly distributed harmonic lateral load. The effect of structural parameters such as thermal constant and taper constant with different aspect ratios on vibration of simply supported-free-simply supported-free plate for maximum deflection for the different values of the fundamental frequency of vibration is studied. Results are presented in tabular form.

1. INTRODUCTION

Non-linear thermal vibration for mechanical device has been one of the hot research topics. A forced vibration device with variable thickness and non-linear thermal effect led by harmful vibration reduction is undoubtedly more economical means. The study of forced vibration behaviour of plates in the presence of non-linear thermal gradient for rectangular plates is required due to its practical importance because machines very often operate under varied temperature conditions. The temperature effects are overlooked in majority of cases though this may be a major concern for the reason that during heated up periods, structures are exposed to high intensity heat fluxes and the material properties undergo significant changes; hence, the thermal effect on modulus of elasticity of material cannot be neglected. In modern times, such materials find application because of their reduction of weight and size, low expenses, and enhancement in effectiveness and strength.

Designing and fabrication of various structures requires accuracy and perfection technically as well as economically. Though plates of variable thickness are frequently used, best fitted and suitably tapered structures are appreciated by structural engineers. It is also important for engineers to find a suitable type of tapering for their mechanical structure as variety of tapered plates are required for various types of structures and for different purposes. Therefore, the thickness variation of the plates plays a vital role in the manufacturing of the structure. Further, tapering saves weight by removing unnecessary weight of the structure. Thickness tapering is desirable since stresses tend to vary significantly within the structure.

The materials are developed depending upon the requirement and durability so that these can be used to give better strength, flexibility, weight effectiveness, and efficiency. Thus, so some new materials and alloys are utilized in making structural parts of equipment used in modern technological industries like space craft, jet engine, earth quake resistance structures, telephone industry etc. It is important that first few frequencies of structure be known before finalizing the design of a structure. The study of vibration of plate structures is important in a wide variety of applications in engineering design. Elastic plates are widely employed nowadays in civil, aeronautical, and marine structural designs. Complex shapes with variety of thickness variation are sometimes incorporated to reduce costly material, lighten the loads, provide ventilation, and alter the resonant frequencies of the structures.

In the recent past, a considerable amount of work has been done on vibration of plates having variable thickness, owing to their continually increasing use in the dynamic design of various engineering structures. However, no work is available on the vibration of rectangular plates of thickness variable with non-linear temperature in the presence of external force. So far only few papers have been devoted to vibration of rectangular plate of variable thickness in presence of external force. Vibration of plates of various shapes, homogeneous, orthotropic or isotropic, with or without variation in thickness, have been studied by various authors, with or without considering the effect of external force.

Akiyama and Kuroda[1] discussed the fundamental frequencies of rectangular plates with linearly varying thickness. Bambill et al.[2] have studied the transverse vibrations of rectangular, trapezoidal, and triangular orthotropic cantilever plates. Civalek[3] discussed the fundamental frequency of isotropic and orthotropic rectangular plates with linearly varying thickness by discrete singular convolution method. Grossi and Laura[4] discussed the transverse vibrations of circular plates of linearly varying thicknesses. Gupta et al.[5,6] studied the thermal effect on vibration of parallelogram plate of linearly varying thickness and bi-directional linearly varying thickness. Gupta et al.[7] studied the thermal gradient effect on vibration of a non-homogeneous orthotropic rectangular plate having bi-directional linearly thickness variation. Gupta et al.[8] did the vibration analysis of visco-elastic rectangular plate with thickness varying linearly in one and parabolically in other direction. Gupta et al.[9] did the vibration study of visco-elastic parallelogram plate of linearly varying thickness. Gupta and Kaur[10] studied the effect of thermal gradient on free vibration of clamped visco-elastic rectangular plates with linear thickness variation in both directions. Gupta and Khanna[11] studied the vibration of visco-elastic rectangular plate with linear thickness variations in both directions. Gupta et al.[12] observed the thermal effect on vibration of non-homogeneous orthotropic rectangular plate having bi-directional parabolically varying thickness. Gupta and Ansari[13] studied the effect of elastic foundation on asymmetric vibration of polar orthotropic linearly tapered circular plates. Lal et al.[14] discussed the Chebyshev polynomials in the study of vibrations of non-uniform rectangular plates. . Laura and Duran[15] wrote a technical note on forced vibrations of a clamped rectangular plate. Laura and Gutierrez[16] carried out the vibration analysis on a rectangular plate subjected to a thermal gradient. Liew and Lim[17] have studied the free transverse vibration analysis of symmetric trapezoidal plates with linearly varying thickness. Liew and Lam[18] worked on the vibration response of symmetrically laminated trapezoidal composite plates with point constraints. Transverse vibrations of clamped trapezoidal plates having rectangular orthotropy were studied by Narita et al.[19] Oniszczuk[20] studied the forced transverse vibrations of an elastically connected complex rectangular simply supported double-plate system.

Qatu[21] presents the natural frequencies for laminated composite angle-ply triangular and trapezoidal plates with completely free boundaries. Qatu et al.[22] have worked on the natural frequencies of trapezoidal plates with completely free boundaries. Raju[23] studied the vibration of thin elastic plates of linearly variable thickness. Saha et al.[24] discussed the nonlinear free vibration analysis of square plates with various boundary conditions. Sakata and Sakata[25] studied the forced vibrations of a non-uniform thickness rectangular plate with two free sides. Sakata[26] discussed the forced vibration of a rectangular plate with non-uniform thickness. Tomar and Gupta[27,28] studied the effect of thermal gradient on frequencies of orthotropic rectangular plate of variable thickness in one and two directions.

Here vibrations of rectangular plate of linearly varying thickness with non-linear temperature in the presence of external force have been studied. The thickness of the plate is assumed to vary linearly along x axis. The plate is simply supported along two opposite sides and is free along the other sides. Approximate formulae are proposed for estimating the maximum deflection of a rectangular plate subject to a uniformly distributed harmonic lateral load. Maximum deflection for the different values of fundamental frequency of vibration is computed for a simply supported-free-simply supported-free plate for various values of thermal constant, taper constant, and aspect ratios. Numerical results are shown in tabular form.

2. EQUATION OF MOTION

Consider a plate that is symmetrical in nature and of rectangular form. Let a be the length of the plate and b be the width of the plate. The rectangular plate to be dealt with here is simply supported along the two opposite sides $y = 0$ and $y =$ b, and is free along the remaining sides $x = 0$ and $x =$ a.

Assuming that the plate under consideration is subjected to steady one dimensional temperature distribution along the length, one can take T as

$$T = T_0\left(1 - \frac{1}{2}\left(\frac{x}{a} + \frac{x^2}{a^2}\right)\right); \tag{1}$$

where T denotes the temperature excess above the reference temperature at any point at distance x/a and T_0 denotes the temperature excess above reference temperature at the end i.e. $x = 0$.

Temperature dependence of the modulus of elasticity for most of engineering materials can be expressed in the form[29]

$$E = E_0\left(1 - \gamma T\right); \tag{2}$$

where E_0 is the value of the Young's modulus at reference temperature i.e. $T = 0$ and γ is the slope of the variation of E with T. Thus, modulus variation becomes

$$E = E_0\left(1 - \alpha\left(1 - \frac{1}{2}\left(\frac{x}{a} + \frac{x^2}{a^2}\right)\right)\right); \tag{3}$$

where $\alpha = \gamma T_0$ $(0 \leq \alpha < 1)$, a parameter known as thermal constant.

The thickness h of the plate at an arbitrary point is assumed to vary linearly in x-direction as

$$h = h_0\left(1 + \beta\frac{x}{a}\right); \tag{4}$$

where h_0 is the thickness of the plate at the side $x = 0$, and β is a constant know as taper constant. The governing differential equation of motion of an isotropic elastic plate of variable thickness for forced vibration in Cartesian coordinates is

$$\nabla^2\left(D\nabla^2 w\right) - (1-\nu)\left(D_{,yy}w_{,xx} - 2D_{,xy}w_{,xy} + D_{,xx}w_{,yy}\right) + \rho h w_{,tt} = q; \tag{5}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a two-dimensional Laplacian operator, $D = \frac{Eh^3}{12(1-\nu^2)}$ is a flexural rigidity, w is deflection, h is thickness of plate, E is modulus of elasticity, t is time, ρ is mass density per unit volume, ν is Poisson's ratio (which is taken constant for simplicity otherwise equation become more critical), and q is lateral load.

A comma followed by a suffix denotes partial differentiation with respect to that variable.

Since thickness varies in x-direction only, so flexural rigidity D of the plate becomes a function of x only. As plate undergoing free vibration with radian frequency ω, so one has

$$w(x, y, t) = W(x, y)\sin\omega t; \tag{6a}$$

$$q(x, y, t) = Q(x, y)\sin\omega t. \tag{6b}$$

Using Eqs. (3), (4), and (6) in Eq. (5), differential equation governing the forced vibration of the plate when it is subjected to a harmonic lateral load comes out as

$$C\left[\left(\frac{\partial^4 W}{\partial x^4}\right)+2\left(\frac{\partial^4 W}{\partial x^2\partial y^2}\right)+\left(\frac{\partial^4 W}{\partial y^4}\right)\right]$$
$$+2\left(\frac{\partial C}{\partial x}\right)\left[\left(\frac{\partial^2 W}{\partial x^3}\right)+\left(\frac{\partial^3 W}{\partial x\partial y^2}\right)\right]$$
$$+\left(\frac{\partial^2 C}{\partial x^2}\right)\left[\left(\frac{\partial^2 W}{\partial x^2}\right)+\nu\left(\frac{\partial^2 W}{\partial y^2}\right)\right]$$
$$-\left(\rho h_0\omega^2\left(1+\frac{\beta x}{a}\right)\frac{W}{D_0}\right)=\frac{Q}{D_0}; \qquad (7)$$

where D_0 is the flexural rigidity at the side $x = 0$ and is given by $E_0h_0^3/12(1-\nu^2)$ and $C = \left(1-\alpha\left(1-\frac{1}{2}\left(\frac{x}{a}+\frac{x^2}{a^2}\right)\right)\right)\left(1+\beta\frac{x}{a}\right)^3$.

3. SOLUTION

The solution of Eq. (7) is expressed as a double trigonometric series,

$$W=\sum_{k=1}^{K}\sum_{l=1}^{L}W_{kl}\left[\sin\left(\frac{k\pi x}{a}\right)+C_{kl1}x^3+C_{kl2}x^2+\right.$$
$$\left.+C_{kl3}x+C_{kl4}\right]\sin\left(\frac{l\pi y}{b}\right). \qquad (8)$$

Here, the deflection W satisfies the boundary conditions

$$W=\frac{\partial^2 W}{\partial y^2}=0 \quad \text{on} \quad y=0,b. \qquad (9)$$

The constants $C_{kli}(i = 1, 2, 3, 4)$ are determined when the deflection satisfies the boundary conditions

$$\left(\frac{\partial^2 W}{\partial x^2}\right)+\nu\left(\frac{\partial^2 W}{\partial y^2}\right)=\left(\frac{\partial^3 W}{\partial x^3}\right)+(2-\nu)\left(\frac{\partial^3 W}{\partial x\partial y^2}\right)$$
$$=0 \quad \text{on} \quad x=0,a. \qquad (10)$$

The lateral load $Q(x,y)$ can be expanded as

$$Q(x,y)=\sum_{k=1}^{K}\sum_{l=1}^{L}Q_{kl}\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right). \qquad (11)$$

Substituting Eqs. (8) and (11) into Eq. (7) gives

$$\sum_{k=1}^{K}\sum_{l=1}^{L}W_{kl}\left[F_{kl1}-\left(\frac{\rho h_0\omega^2}{D_0}\right)F_{kl2}\right]\sin\left(\frac{l\pi y}{b}\right)=$$
$$=\sum_{k=1}^{K}\sum_{l=1}^{L}\left(\frac{Q_{kl}}{D_0}\right)\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{l\pi y}{b}\right); \qquad (12)$$

where the quantities F_{kli} $(i = 1, 2)$ are functions of x, β, α and a/b only. Hence, it can be expanded in trigonometric series form as

$$F_{kli}=\sum_{m=1}^{K}B_{klmi}\sin\left(\frac{m\pi x}{a}\right). \qquad (13)$$

Substituting Eq. (13) into Eq. (12) then gives

$$\sum_{k=1}^{K}W_{kl}\left[B_{klm1}-\left(\frac{\rho h_0\omega^2}{D_0}\right)B_{klm2}\right]=\frac{Q_{ml}}{D_0},$$
$$m=1,2,\ldots,K; \qquad (14)$$

Table 1. Values of the maximum deflection W_{max} for different values of taper constant β and for fixed value of aspect ratio $a/b = 1.0$, thermal constant $\alpha = 0.5$, $x/a = 0.1$, and $y/b = 0.5$.

β	W_{max}			
	$\omega = 0.0$	$\omega = 0.3\,\omega_0$	$\omega = 0.5\,\omega_0$	$\omega = 0.8\,\omega_0$
0.0	0.01120	0.01280	0.01635	0.02084
0.2	0.01227	0.01360	0.01700	0.02155
0.4	0.01301	0.01436	0.01765	0.02228
0.6	0.01377	0.01517	0.01844	0.02289
0.8	0.01458	0.01588	0.01918	0.02363
1.0	0.01523	0.01653	0.02082	0.02411

Table 2. Values of the maximum deflection W_{max} for different values of thermal constant α and for fixed value of aspect ratio $a/b = 1.0$, taper constant $\beta = 0.5$, $x/a = 0.1$, and $y/b = 0.5$.

α	W_{max}			
	$\omega = 0.0$	$\omega = 0.3\,\omega_0$	$\omega = 0.5\,\omega_0$	$\omega = 0.8\,\omega_0$
0.0	0.01708	0.01825	0.02196	0.02635
0.2	0.01530	0.01671	0.02038	0.02475
0.4	0.01378	0.01544	0.01910	0.02351
0.6	0.01226	0.01441	0.01783	0.02235
0.8	0.01072	0.01335	0.01658	0.02120
1.0	0.00966	0.01234	0.01542	0.01992

for each l. Thus, one can compute the deflection W satisfying both the governing Eq. (7) and the boundary conditions, Eqs. (9) and (10), by substituting the coefficients W_{kl} obtained from Eq. (14) into Eq. (8).

4. RESULTS AND DISCUSSION

Numerical calculations were performed for thermally forced rectangular plate subjected to a uniformly distributed lateral load $Q(x,y) = Q_0$ with number of term in series for K and L as 60. Poisson ratio ν is taken as 0.3. Let the fundamental natural frequency and maximum deflection in x and y direction be denoted by ω_0 and W_{max}, respectively. Maximum deflection W_{max} is given by $W/(Q_0a^4/D_0)$ and calculated for four typical case $\omega = 0.0$, $0.3\ \omega_0$, $0.5\ \omega_0$, and $0.8\ \omega_0$ for different values of taper constant β, thermal constant α and aspect ratio a/b with two combinations of x/a and y/b for simply supported-free-simply supported-free rectangular plate of linearly variable thickness with thermal constant. Results are displayed in Tables 1–6.

Table 1 displays variation of maximum deflection W_{max} with taper constant β. Thermal constant has to be taken as $\alpha = 0.5$ for aspect ratios a/b = 1.0 and $x/a = 0.1$, $y/b = 0.5$. It has been observed from the table that when taper constant increases, maximum deflection also increases for all the four values of fundamental frequency. It is interesting to note that as fundamental frequency increases, maximum deflection also increases.

Table 2 contains numerical results for maximum deflection W_{max} verses thermal constant α. Value of taper constant $\beta = 0.5$ has been taken for aspect ratios $a/b = 1.0$ and $x/a = 0.1$, $y/b = 0.5$ for all the four values of fundamental frequency. It is clear from the table that when thermal constant increases, maximum deflection decreases. Also, when fundamental frequency increases, maximum deflections increase.

Table 3 show variation of maximum deflection with aspect ratio. A combination of taper constant and thermal constant

Table 3. Values of the maximum deflection W_{max} for different values of aspect ratio a/b and for fixed value of taper constant $\beta = 0.5$, thermal constant $\alpha = 0.5$, $x/a = 0.1$, and $y/b = 0.5$.

a/b	W_{max}			
	$\omega = 0.0$	$\omega = 0.3\,\omega_0$	$\omega = 0.5\,\omega_0$	$\omega = 0.8\,\omega_0$
0.5	0.01432	0.01599	0.01925	0.02380
1.0	0.01335	0.01479	0.01810	0.02265
1.5	0.01246	0.01359	0.01695	0.02154
2.0	0.01138	0.01245	0.01540	0.02053
2.5	0.01031	0.01132	0.01430	0.01945
3.0	0.00925	0.01028	0.01316	0.01839

Table 4. Values of the maximum deflection W_{max} for different values of taper constant β and for fixed value of aspect ratio $a/b = 1.0$, thermal constant $\alpha = 0.5$, $x/a = 0.5$, and $y/b = 0.1$.

β	W_{max}			
	$\omega = 0.0$	$\omega = 0.3\,\omega_0$	$\omega = 0.5\,\omega_0$	$\omega = 0.8\,\omega_0$
0.0	0.01112	0.01269	0.01623	0.02073
0.2	0.01219	0.01349	0.01689	0.02145
0.4	0.01291	0.01424	0.01752	0.02217
0.6	0.01367	0.01507	0.01832	0.02278
0.8	0.01457	0.01578	0.01907	0.02352
1.0	0.01513	0.01643	0.02072	0.02402

Table 5. Values of the maximum deflection W_{max} for different values of thermal constant α and for fixed value of aspect ratio $a/b = 1.0$, taper constant $\beta = 0.5$, $x/a = 0.5$, and $y/b = 0.1$.

α	W_{max}			
	$\omega = 0.0$	$\omega = 0.3\,\omega_0$	$\omega = 0.5\,\omega_0$	$\omega = 0.8\,\omega_0$
0.0	0.01702	0.01818	0.02187	0.02624
0.2	0.01524	0.01663	0.02029	0.02465
0.4	0.01371	0.01535	0.01900	0.02341
0.6	0.01219	0.01430	0.01773	0.02226
0.8	0.01065	0.01325	0.01649	0.02111
1.0	0.00958	0.01224	0.01533	0.01984

has been considered with $x/a = 0.1$, $y/b = 0.5$ for all the four values of fundamental frequency. It has been observed that when aspect ratio increases, maximum deflection decreases for all the four cases of fundamental frequency.

Table 4 contains variation of maximum deflection W_{max} with taper constant β. Thermal constant has to be taken as $\alpha = 0.5$ for aspect ratios $a/b = 1.0$ and $x/a = 0.5$, $y/b = 0.1$. It has been observed from the table that when taper constant increases, maximum deflection also increases for all the four values of fundamental frequency. It is interesting to note that as fundamental frequency increases, maximum deflection also increases.

Table 5 contains numerical results for maximum deflection W_{max} verses thermal constant α. One value of taper constant $\beta = 0.5$ has been taken for aspect ratios $a/b = 1.0$ and $x/a = 0.5$, $y/b = 0.1$ for all the four values of fundamental frequency. It is clear from the table that when thermal constant increases, maximum deflection decreases. Also, when fundamental frequency increases, maximum deflections increase.

Table 6 contains variation of maximum deflection with aspect ratio. A combination of taper constant and thermal constant has been considered with $x/a = 0.5$, $y/b = 0.1$ for all the four values of fundamental frequency. It has been observed that when aspect ratio increases, maximum deflection decreases for all the four cases of fundamental frequency. It is evident from the tables that on increasing the values of x/a and decreasing value of y/b, maximum deflection decreases.

Table 6. Values of the maximum deflection W_{max} for different values of aspect ratio a/b and for fixed value of taper constant $\beta = 0.5$, thermal constant $\alpha = 0.5$, $x/a = 0.5$, and $y/b = 0.1$.

a/b	W_{max}			
	$\omega = 0.0$	$\omega = 0.3\,\omega_0$	$\omega = 0.5\,\omega_0$	$\omega = 0.8\,\omega_0$
0.5	0.01426	0.01591	0.01915	0.02368
1.0	0.01328	0.01471	0.01800	0.02253
1.5	0.01235	0.01350	0.01684	0.02143
2.0	0.01129	0.01236	0.01529	0.02042
2.5	0.01023	0.01122	0.01418	0.01935
3.0	0.00916	0.01018	0.01304	0.01829

5. CONCLUSIONS

It can be concluded from the results that deflection increases with increase in taper constant and decreases with increase in thermal constant sharply as compared to linear temperature. Also, when aspect ratio a/b increases, deflection decrease. It is also clear from the tables that when y/b is more in comparison to x/a, then deflection is large.

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