Analytical and Numerical Solution of 2D Problem for Transversely Isotropic Generalized Thermoelastic Medium with Green-Naghdi Model II

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In this paper, a comparison was made between the analytical and numerical solution of a two-dimensional problem for a transversely isotropic generalized thermoelastic medium. The study is carried out in the context of generalized thermoelasticity proposed by Green and Naghdi’s theory of type II. The problem has been solved analytically using the normal mode method with the eigenvalue approach and numerically using a finite element method. The accuracy of the finite element formulation was validated by comparing the analytical and numerical solutions for the field quantities.

1. INTRODUCTION

Thermoelasticity theories, which admit a finite speed for thermal signals (second sound), have aroused much interest in the last three decades. In contrast to the conventional coupled thermoelasticity theory based on a parabolic heat equation,\(^1\)–\(^3\) which predicts an infinite speed for the propagation of heat, these theories involve a hyperbolic heat equation and are referred to as generalized thermoelasticity theories. Among these generalized theories, the theory proposed by Lord and Shulman\(^4\) involving one relaxation time and the theory developed by Green and Lindsay\(^5\) involving two relaxation times have been subjected to a large number of investigations. In the Lord–Shulman theory, a modified Fourier’s law of heat conduction including both the heat and its time derivative replaces the conventional Fourier’s law whereas in the Green-Lindsay model, Fourier’s law of heat conduction is left unchanged but the classical energy equation and stress-strain temperature relations are modified. Verma and Hasebe\(^6\) studied wave propagation in transversely isotropic plates in the context of generalized thermoelasticity proposed by the Lord-Shulman theory. Othman\(^7\) studied the dependence of the modulus of elasticity on the reference temperature in a two-dimensional problem of generalized thermoelasticity under Lord-Shulman theory. In view of some experimental evidence available in favour of finiteness of heat propagation speed, generalized thermoelasticity theories are considered to be more realistic than the conventional thermoelasticity theory in dealing with practical problems involving very large heat fluxes and short time inter-
mits the dissipation of energy. In this model, the constitutive
equations are derived by starting with the reduced energy
equation, where the thermal displacement gradient in addition
to temperature gradient are among the constitutive variables.
In the context of the linearized version of this theory,\textsuperscript{20} the
theorem on uniqueness of solutions has been established by
Chandrasekharaiah.\textsuperscript{22,23} Othman and Atwa\textsuperscript{24} studied the ef-
effect of magnetic fields on a two-dimensional problem of gen-
eralized thermoelasticity without energy dissipation. Othman
and Atwa\textsuperscript{25} have also studied the response of micropolar ther-
moelastic medium with voids due to various sources with and
without energy dissipation. The problems have been solved by
applying the eigenvalue approach. Roychoudhuri and Dutta\textsuperscript{26}
studied thermoelastic interactions in an isotropic homogeneous
thermoelastic solid containing time-dependent distributed heat
sources which vary periodically for a finite time interval in the
context of TEWOED. Problems concerning generalized ther-
moelasticity proposed by Green and Naghdi’s theory of both
types II and III have been studied by many other authors.\textsuperscript{27,28}

The exact solution of the generalized thermoelasticity the-
ory governing equations for a coupled and non-linearinear exists
only for very special and simple initial and boundary
problems. In view of calculating general problems, a numeri-
ical solution technique is to be used. For this reason, the finite
element method (FEM) is chosen. The method of weighted
residuals offers us the formulation of the finite element equa-
tions and we obtain best approximated solutions to linear and
nonlinear ordinary and partial differential equations. Applying
this method basically involves three steps. The first step is to
assume the general behavior of the unknown field variables
in such a way as to satisfy the given differential equations.
Substitution of these approximating functions into the differen-
tial equations and boundary conditions results in some errors,
called the residual. This residual has to vanish in an average
sense over the solution domain. The second step is the time in-
TEGRATION. The time derivatives of the unknown variables have
to be determined by former results. The third step is to solve the
equations resulting from the first and the second step using the
solving algorithm of the finite element program.\textsuperscript{29} Abbas
and his colleagues\textsuperscript{30–39} applied the FEM in different problems.

In the present paper, we have formulated the problem of a
two-dimensional problem for a transversely isotropic general-
ized thermoelasticity under the Green-Naghdi theory of type II.
The problem has been solved analytically using a normal
mode method with the eigenvalue approach and numerically
using a FEM. The results for a transversely isotropic material
have been deduced numerically and presented graphically to
compare with those of transversely isotropic material which
was obtained analytically.

2. FORMULATION OF THE PROBLEM

We consider the problem of a transversely isotropic ther-
moelastic half-space ($x \geq 0$). The surface of the half-space
is subjected to a thermal shock, which is a function of $y$ and $t$.
Thus, all the quantities considered will be functions of the time
variable $t$, and of the coordinates $x$ and $y$. Then the displace-
ment vector $\mathbf{u}$ and temperature $T$ can be taken in the following
form

$$
\mathbf{u} = (u(x, y, t), v(x, y, t), 0), \quad T = T(x, y, t).
$$

Now the constitutive relations in the present case are

$$
\begin{align*}
\sigma_{xx} &= C_{11}e_{xx} + C_{12}e_{yy} - \beta_{11}(T - T_0); \\
\sigma_{yy} &= C_{12}e_{xx} + C_{11}e_{yy} - \beta_{11}(T - T_0); \\
\sigma_{xy} &= (C_{11} - C_{12})e_{xy};
\end{align*}
$$

The equations of motion along $x$ and $y$ directions can be ob-
tained as follows

$$
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2}; \\
\frac{\partial \sigma_{yy}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}; \\
K^*(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) &= \rho C_E \frac{\partial^2 T}{\partial t^2} + \beta_{11}T_0 \frac{\partial^2 e}{\partial t^2};
\end{align*}
$$
or,

$$
\begin{align*}
C_{11} \frac{\partial^2 u}{\partial x^2} + C_{11} - C_{12} \frac{\partial^2 u}{\partial y^2} + \\
\frac{C_{11} + C_{12}}{2} \frac{\partial^2 u}{\partial x \partial y} - \beta_{11} \frac{\partial T}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2}; \\
C_{11} - C_{12} \frac{\partial^2 v}{\partial x^2} + C_{12} \frac{\partial^2 v}{\partial y^2} + \\
\frac{C_{11} + C_{12}}{2} \frac{\partial^2 v}{\partial x \partial y} - \beta_{11} \frac{\partial T}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}; \\
K^*(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) &= \rho C_E \frac{\partial^2 T}{\partial t^2} + \beta_{11}T_0 \frac{\partial^2 e}{\partial t^2};
\end{align*}
$$

where $e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, $e_{xx} = \frac{\partial u}{\partial x}$, $e_{yy} = \frac{\partial v}{\partial y}$, $e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\beta_{11}$ is the stress temperature coefficient, $C_E$ is the specific heat
at constant volume, and $K^*$ is the additional material constant
for Green-Naghdi theory of type II.

The following nondimensional variables are introduced:

$$
(x', y') = \frac{1}{L}(x, y), \quad t' = \frac{c_1}{L} t,
$$

$$
(u', v') = \frac{C_{11}}{C_{111140}}(u, v), \quad T' = \frac{\beta_{11}(T - T_0)}{C_{111140}}.
$$

and in terms of the non-dimensional quantities defined in
Eq. (11), the above governing equations reduce to (dropping the
dashed for convenience)

$$
\begin{align*}
\sigma_{xx} &= \frac{\partial u}{\partial x} + a_1 \frac{\partial v}{\partial y} - T; \\
\sigma_{yy} &= a_1 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - T; \\
\sigma_{xy} &= a_2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right);
\end{align*}
$$

$$
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= \frac{\partial^2 u}{\partial t^2}; \\
\frac{\partial \sigma_{yy}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial y} &= \frac{\partial^2 v}{\partial t^2}; \\
\epsilon_1 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) &= \frac{\partial^2 T}{\partial t^2} + \epsilon_2 \frac{\partial^2 e}{\partial t^2}.
\end{align*}
$$
where \( a_1 = \frac{C_{12}}{C_{11}}, \quad a_2 = \frac{C_{44}-C_{12}}{C_{11}}, \quad \epsilon_1 = \frac{K^*}{\rho_c E c_1^2}, \quad \epsilon_2 = \frac{\nu_1}{\rho_c E c_1^2}. \)

### 3. NORMAL MODE ANALYSIS

The solution of the considered physical variable can be decomposed in terms of normal modes as the following form:

\[
[u, v, T, \sigma_{ij}](x, y, t) = [u^*, v^*, T^*, \sigma_{ij}^*](x) \exp(\omega t + i b y);
\]

where \( \omega \) is a complex constant, \( i = \sqrt{-1} \), \( b \) is the wave number in the \( y \) direction and \( u^*(x), v^*(x), T^*(x) \) and \( \sigma_{ij}^*(x) \) are the amplitudes of the field quantities. Subsisting from Eq. (18) in Eqs. (12)–(17), we get

\[
\sigma_{xx}^* = \frac{du^*}{dx} + i b u^* - T^*; \quad \sigma_{yy}^* = a_1 \frac{du^*}{dx} + i b v^* - T^*; \quad \sigma_{xy}^* = a_2 \left( i b u^* + \frac{du^*}{dx} \right); \quad e^* = \frac{du^*}{dx} + i b v^*; \\
\frac{d^2 u^*}{dx^2} = B_{41} u^* + B_{45} \frac{du^*}{dx} + B_{46} \frac{dT^*}{dx}; \quad \frac{d^2 v^*}{dx^2} = B_{52} v^* + B_{53} T^* + B_{54} \frac{du^*}{dx}; \quad \frac{d^2 T^*}{dx^2} = B_{62} v^* + B_{63} T^* + B_{64} \frac{du^*}{dx};
\]

where

\[
B_{41} = \beta^2 a_2 + \omega^2, \quad B_{45} = -i b, \quad B_{46} = 1, \quad B_{52} = \frac{\nu_1}{\nu_2}(\beta^2 + \omega^2), \quad B_{53} = \frac{\nu_1}{\nu_2}, \quad B_{54} = \frac{\nu_1}{\nu_2} \beta^2, \quad B_{62} = \beta^2 + \frac{\omega^2}{\nu_1}, \quad B_{63} = \frac{\omega^2}{\nu_1};
\]

Equations (23)–(25) can be written in a vector-matrix differential equation as follows:

\[
\frac{d\vec{V}}{dx} = \mathbf{B}\vec{V};
\]  

where \( \vec{V} = [u^* \quad v^* \quad T^* \quad \frac{du^*}{dx} \quad \frac{dv^*}{dx} \quad \frac{dT^*}{dx}]^T \) and

\[
\mathbf{B} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
B_{41} & 0 & 0 & 0 & B_{45} & B_{46} \\
0 & B_{52} & B_{53} & B_{54} & 0 & 0 \\
0 & B_{62} & B_{63} & B_{64} & 0 & 0
\end{bmatrix}.
\]

### 4. SOLUTION OF THE VECTOR-MATRIX DIFFERENTIAL EQUATION

Let us now proceed to solve Eq. (26) by the eigenvalue approach proposed. The characteristic equation of the matrix \( \mathbf{B} \) takes the form

\[
\lambda^6 - \eta_1 \lambda^4 + \eta_2 \lambda^2 + \eta_3 = 0;
\]

where \( \eta_1 = B_{41} + B_{52} + B_{45} B_{53} + B_{46} B_{64}, \quad \eta_2 = B_{41} B_{52} - B_{53} B_{62} - B_{45} B_{54} B_{63} + B_{46} B_{54} B_{63} + B_{45} B_{54} B_{63} + B_{46} B_{54} B_{63} + B_{45} B_{54} B_{63}, \quad \eta_3 = B_{41} B_{53} B_{62} - B_{45} B_{54} B_{63}. \) The roots of the characteristic Eq. (27), which are also the eigenvalues of matrix \( \mathbf{B} \), are of the form \( \lambda = \pm \lambda_1, \quad \lambda = \pm \lambda_2, \quad \lambda = \pm \lambda_3. \) The eigenvector \( \vec{X} = [x_1, x_2, x_3, x_4, x_5, x_6]^T \), corresponding to eigenvalue \( \lambda \), can be calculated as

\[
x_1 = -\lambda \left( B_{45} B_{63} + B_{46} \left( -B_{52} + \lambda^2 \right) \right); \quad x_2 = B_{41} \left( B_{63} + B_{46} \lambda \right) \lambda^2; \quad x_3 = -B_{41} B_{52} + \left( B_{41} + B_{52} + B_{45} B_{63} \right) \lambda^2 - \lambda^4; \quad x_4 = \lambda x_1; \quad x_5 = \lambda x_2; \quad x_6 = \lambda x_3.
\]

From Eqs. (28)–(31) we can easily calculate the eigenvector \( \vec{X}_j \), corresponding to eigenvalue \( \lambda_j, \ j = 1, 2, 3, 4, 5, 6 \). For further reference, we shall use the following notations:

\[
\vec{X}_1 = [\vec{X}]_{\lambda=-\lambda_1}, \quad \vec{X}_2 = [\vec{X}]_{\lambda=-\lambda_2}, \quad \vec{X}_3 = [\vec{X}]_{\lambda=-\lambda_3}, \quad \vec{X}_4 = [\vec{X}]_{\lambda=\lambda_1}, \quad \vec{X}_5 = [\vec{X}]_{\lambda=\lambda_2}, \quad \vec{X}_6 = [\vec{X}]_{\lambda=\lambda_3}.
\]

The solution of Eq. (26) can be written as follows:

\[
\vec{V} = \sum_{j=1}^{6} A_j \vec{X}_j e^{\lambda_j x} \]

where the terms containing exponentials of growing nature in the space variable \( x \) have been discarded due to the regularity condition of the solution at infinity, \( A_1, A_2, \) and \( A_3 \) are constants to be determined from the boundary condition of the problem. Thus, the field variables can be written for \( x \geq 0, \ t > 0, \ -\infty \leq y \leq \infty \) as:

\[
u(x, y, t) = e^{(\omega t + ib y)} \sum_{j=1}^{3} A_j x_4^j e^{-\lambda_j x}; \quad (34)
\]

\[
v(x, y, t) = e^{(\omega t + ib y)} \sum_{j=1}^{3} A_j x_5^j e^{-\lambda_j x}; \quad (35)
\]

\[
T(x, y, t) = e^{(\omega t + ib y)} \sum_{j=1}^{3} A_j x_6^j e^{-\lambda_j x}; \quad (36)
\]

To complete the solution, we have to know the constants \( A_1, A_2, \) and \( A_3 \), so we will use the following boundary conditions.

### 5. APPLICATION

1. The mechanical boundary condition that the bounding plane to the surface \( x = 0 \) has no traction, so we have

\[
\sigma_{xx}(0, y, t) = \sigma_{xy}(0, y, t) = 0. \quad (37)
\]

2. The thermal boundary condition is

\[
vT - \frac{\partial T}{\partial x} = r(0, y, t); \quad (38)
\]

where \( \frac{\partial T}{\partial x} \) denotes the normal components of the heat flux vector, \( v \) is the Biot’s number, and \( r(0, y, t) \) represents the intensity of the applied heat sources. From the boundary conditions Eqs. (37) and (38), we can obtain

\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix} = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0 \\
r^*
\end{bmatrix} . \quad (39)
\]
where \( X_{11} = -\lambda_1 x_1^2 + iba_1 x_3^2 - x_6^2, X_{12} = -\lambda_2 x_1^2 + iba_1 x_5^2 - x_6^2, X_{13} = -\lambda_3 x_1^2 + iba_3 x_3^2 - x_6^2, X_{21} = ibx_1^2 - \lambda_1 x_3^2, X_{22} = ibx_1^2 - \lambda_2 x_5^2, X_{23} = ibx_1^2 - \lambda_3 x_3^2, X_{31} = (v + \lambda_1) x_6, X_{32} = (v + \lambda_2) x_6, X_{33} = (v + \lambda_3) x_6. \)

6. FINITE ELEMENT METHOD

The finite element method is a powerful technique originally developed for numerical solutions of complex problems in structural mechanics, and it remains the method of choice for complex systems. A further benefit of this method is that it allows physical effects to be visualized and quantified regardless of experimental limitations. In this section, the governing equations of the two-dimensional problem for transversely isotropic generalized thermoelastic medium without energy dissipation are summarized, using the corresponding finite element equations. In the FEM, the displacement components \( u, v \) and temperature \( T \) are related to the corresponding nodal values by

\[
\begin{align*}
   u &= \sum_{i=1}^{m} N_i u_i(t), \\
v &= \sum_{i=1}^{m} N_i v_i(t), \\
   T &= \sum_{i=1}^{m} N_i T_i(t),
\end{align*}
\]

where \( m \) denotes the number of nodes per element, and \( N \) the shape functions. In the framework of standard Galerkin procedure, the weighting functions and the shape functions coincide. Thus,

\[
\begin{align*}
   \delta u &= \sum_{i=1}^{m} N_i \delta u_i, \\
   \delta v &= \sum_{i=1}^{m} N_i \delta v_i, \\
   \delta T &= \sum_{i=1}^{m} N_i \delta T_i.
\end{align*}
\]

Thus, the finite element equations corresponding to Eqs. (23)–(25) can be obtained as

\[
\begin{align*}
   \int_{0}^{\infty} \frac{d\delta u^*}{dx} \frac{du^*}{dx} \, dx + \\
   \int_{0}^{\infty} \frac{d\delta v^*}{dx} \frac{dv^*}{dx} \, dx + \\
   \int_{0}^{\infty} \frac{d\delta T^*}{dx} \frac{dT^*}{dx} \, dx + \\
   \int_{0}^{\infty} \delta u^* \left( B_{41} u^* + B_{42} \frac{dv}{dx} + B_{43} \frac{dT}{dx} \right) \, dx = 0, \\
   \int_{0}^{\infty} \delta v^* \left( B_{51} u^* + B_{52} \frac{dv}{dx} + B_{53} \frac{dT}{dx} \right) \, dx = 0, \\
   \int_{0}^{\infty} \delta T^* \left( B_{61} u^* + B_{62} \frac{dv}{dx} + B_{63} \frac{dT}{dx} \right) \, dx = 0.
\end{align*}
\]

7. NUMERICAL RESULTS AND DISCUSSION

With the view of illustrating the numerical results, the material chosen for the plate is magnesium (Mg), the physical data for which is given by the following:\( C_{11} = 5.974 \times 10^{10} \) \( \text{Nm}^{-2}, \ C_{12} = 2.624 \times 10^{10} \) \( \text{Nm}^{-2}, \ C_{22} = 1.04 \times 10^{13} \) \( \text{Jkg}^{-1} \), \( \beta_{11} = 2.17 \times 10^{10} \) \( \text{Nm}^{-2}, \ C_{13} = 6.17 \times 10^{10} \) \( \text{Nm}^{-2}, \ C_{44} = 1.51 \times 10^{10} \) \( \text{Nm}^{-2}, \rho = 1.74 \times 10^{3} \) \( \text{kgm}^{-3}, \ T_0 = 298 \) K, \( \beta_{11} = 2.68 \times 10^{9} \) \( \text{Nm}^{-2} \), \( \beta_{12} = 2.02 \times 10^{-2}, \ K = 200, K_1 = 1.7 \times 10^{2} \) \( \text{Wm}^{-2} \).

Figures 1–6 are drawn to give a comparison of the results for displacements, temperature, and stresses with that of an isotropic material for two methods of solution under the Green-Naghdi theory of type II, against the thickness \( x \) for \( y = 0.25, 1 \) and \( t = 0.2, 0.4 \). Figures 1 and 2 are plotted to show the variation of displacements \( u \) and \( v \) against \( x \) and \( y \) and make a comparison of the results which were obtained by the analytical and numerical solution with isotropic material for the Green-Naghdi model II. The effect of the values of \( y \) and \( t \) is directly proportional with the value of the thickness \( x \) on the distributions for both of the displacement components \( u \) and \( v \), since the distributions are increasing with the increase of the values of \( y \) and \( t \) for \( x > 0 \).

Figure 3 clarifies the variation of temperature \( T \) against \( x \) and \( y \) and study a comparison of the results which were obtained by the analytical and numerical solution with isotropic material for the Green-Naghdi model II. The distribution of \( T \) decreases with the increase of \( y \) and \( t \) values for \( x > 0 \). Figures 4, 5 and 6 are plotted to show the variation of stress components \( \sigma_{xx}, \sigma_{xy}, \sigma_{yy} \) against \( x \) and \( y \) and make a comparison of the results which were obtained by the analytical and numerical solution with isotropic material for the Green-Naghdi model II. The distributions of \( \sigma_{xx} \) and \( \sigma_{yy} \) increase with the increase of \( y \) and \( t \) values for \( x > 0 \), while the distribution of \( \sigma_{xy} \) decreases with the increase of \( y \) and \( t \) values for \( x > 0 \). It is clear that the qualitative behaviour of the analytical solution and the numerical solution seem to be identical for all comparisons, and all the curves converge to zero as \( x \to \infty \).

The value of the error is directly proportional to the intensity of the applied heat source \( r \) for the distributions of both the displacement components \( u \) and \( v \) and the temperature \( T \) where \( 0 < r < 4 \). The value of error lies between \( 10^{-12} \) and \( 10^{-9} \) for the distribution of \( u \), the value of error lies between \( 10^{-11} \) and \( 10^{-9} \) for the distribution of \( v \), and the value of error lies between \( 10^{-13} \) and \( 10^{-11} \) for the distribution of \( T \). As a result of the very small value of error, two solutions obtained by the normal mode analysis method and the FEM seem to be identical.

Finally, after validating the FEM solutions, one can discuss the following comparison in the case, which has its exact solutions. Table 1 shows a comparison of the FEM solution with the exact solution for displacement \( u \). Table 2 shows a comparison of the FEM solution with the exact solution for displacement \( v \). Table 3 shows a comparison of the FEM solution with the exact solution for temperature distributions. It is clear from the results that the FEM is more efficient and accurate.
Figure 1. Horizontal displacement $u$ distribution against $x$.

Figure 2. Vertical displacement $v$ distribution against $x$.

Figure 3. Temperature distribution against $x$. 
Figure 4. Stress component $\sigma_{xx}$ distribution against $x$.

Figure 5. Stress component $\sigma_{xy}$ distribution against $x$.

Figure 6. Stress component $\sigma_{yy}$ distribution against $x$. 
REFERENCES


