Vibration Analysis of Axially Functionally Graded Tapered Euler-Bernoulli Beams Based on Chebyshev Collocation Method

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The bending vibration behavior of a non-uniform axially functionally graded Euler-Bernoulli beam is investigated based on the Chebyshev collocation method. The cross-sectional and material properties of the beam are assumed to vary continuously across the axial direction. The Chebyshev differentiation matrices are used to reduce the ordinary differential equations into a set of algebraic equations to form the eigenvalue problem associated with the free vibration. Some calculated results are compared with numerical results in the published literature to validate the accuracy of the present model. A good agreement is observed. The effects of the taper ratio, volume fraction index, and restraint types on the natural frequency of axially functionally graded beams with non-uniform cross section are examined.

1. INTRODUCTION

Functionally graded (FG) materials are special composites with smoothly varying material properties along any desired spatial direction. These graded properties can be achieved by gradually changing the volume fraction of constituents along a typical direction according to the polynomial, power, and exponential laws. Due to these particular graded properties, the applications of structures with FG materials have received wide attention in civil, automobile and aerospace industries for the past few decades. Because a beam is both a commonly and widely used member in structures and machines, to better understand the dynamic behavior of a beam made of FG materials is necessary from a structural design point of view. In past years, the dynamic behaviors of FG beams with material properties varying along the beam thickness have been thoroughly investigated by many researchers using various analytical and numerical methods.[1–12] It is well-known that beam structures with varying cross-sectional and material properties along the length direction are commonly used in buildings, bridges and mechanical components due to the fact that they are capable of optimizing the strength and weight of the structure. Hence, it is important to accurately predict and evaluate the dynamic characteristics of such beam structures. In this regard, only the literature related to the FG beams with axially varying properties will be discussed next.

Due to the variable coefficients in the governing equation, the dynamic analysis of axially FG beams with tapered cross-sections becomes more mathematically complex. Therefore, the dynamic problems of FG beams with material and cross-sectional properties varying along the axial direction have been largely studied by numerical methods based on various beam theories. Aydogdu investigated the vibration and buckling analysis of simply supported FG Euler-Bernoulli beams with axial variation of material properties based on a semi-inverse method.[13] Huang and Li applied the integral equation method to study the free vibration of non-uniform axially FG Euler-Bernoulli beams.[14] The effects of material graded parameters on natural frequencies of the beams were investigated. Based on the finite element method, Shahba et al. dealt with the free vibration and stability of axially FG tapered Timoshenko beams with respective classic and non-classical boundary conditions.[15] Hein and Feklistova studied the free vibration of non-uniform axially FG beams using the Euler-Bernoulli beam theory and Haar wavelet approach.[16] The results revealed that the Haar wavelet approach was capable of calculating the frequencies of beams with different non-uniform cross-sections, bending rigidity and mass density. Shahba and Rajasekaran studied the free vibration and stability of axially FG tapered Euler–Bernoulli beams by using both the differential transform element method and differential quadrature element method of lowest-order.[17] The free longitudinal and transverse natural frequencies, and the critical buckling loads, of the beams were determined by the two numerical methods. Li et al. presented the exact frequency equations of the free vibration for axially exponentially FG beams with different end conditions based on an analytical approach.[18] Rajasekaran analysed the bending vibration of rotating axially FG tapered Euler–Bernoulli beams based on the differential transform element method and differential quadrature element method of lowest-order.[19] The effects of material property, taper ratio, rotating speed, hub radius and tip mass on the natural frequencies were investigated. Huang et al. investigated the free vibration of non-uniform and axially FG Timoshenko beams with various boundary conditions by using a unified approach.[20] Sarkar and Ganguli studied the free vibration of axially FG uniform Timoshenko beams with fixed–fixed boundary conditions.[21] Tang et al. derived the exact frequency equations of the free transverse vibration of exponentially FG beams with non-uniform cross-section based on the Timoshenko beam theory.[22] Liu et al. analysed the free bending vibration of axially FG Euler-Bernoulli beams with tapered cross-section using the spline finite point method.[23] The effects of the material and cross-sectional properties varying along the axial direction on the natural frequencies were discussed. Cao and Gao investigated the free vibration of axially FG beams with non-uniform cross-section by using the asymptotic development method.[24]

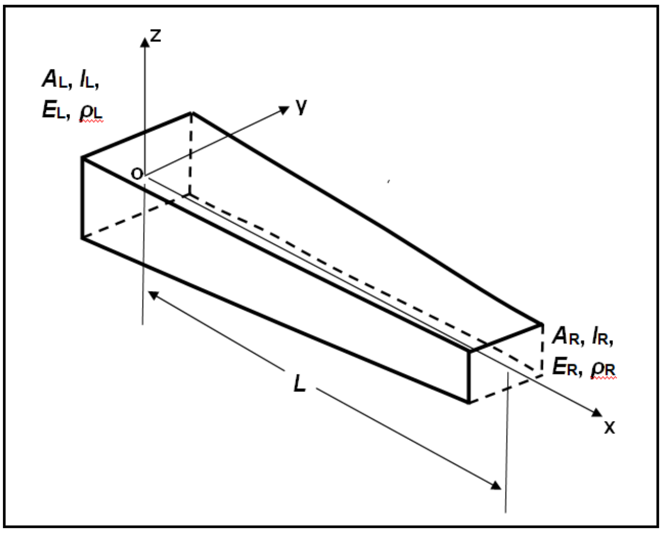

Figure 1. Configuration and coordinates of tapered axially FG beam.

As mentioned above, various analytical and numerical methods had been used to effectively investigate the influences of material and geometric parameters on the dynamic characteristics of axially FG beams. It is well-known that the Chebyshev collocation method has been applied to different mathematical and engineering models because of its high rate of convergence and predictable accuracy.[8–10,12,25–34] However, the application of this method to the bending vibration of axially FG tapered beams has seldom been reported. Hence, the present paper attempts to study the free transverse vibration of the axially FG Euler-Bernoulli beam with a tapered cross-section based on the Chebyshev collocation method. The Chebyshev differentiation matrices are applied to transform the governing differential equations into a set of algebraic eigenvalue equations. The lateral natural frequencies of tapered axially FG beams with various boundary conditions are then obtained by solving the generalized eigenvalue equation. The material properties axially graded according to the exponential, polynomial and power functions are considered. The rectangular cross section is assumed to be tapered linearly in the width and height directions along the beam length. The effects of the taper ratios, axially graded properties, and boundary conditions on the free vibration behaviors of the axially FG tapered beams are investigated. Several numerical results are evaluated and compared with those in the published literature to validate the accuracy of the present model. The results reveal that the proposed method can be used to study the free vibration of tapered Euler-Bernoulli beams with different axially graded material properties described by typical functions under various boundary conditions with good accuracy.

2. PROBLEM FORMULATION

The present paper investigates the bending vibration behavior of non-uniform axially FG beams under various boundary conditions. The material and cross-section properties were assumed to be varied along the longitudinal direction of the beam. Figure 1 illustrates the geometric configuration and coordinate system of the beam with a tapered section along the height and width directions. The origin o of the coordinate xyz was the centre of the left-end plane of the beam. The x-axis, y-axis and z-axis originating from the origin were found in the length, width, and thickness directions, respectively. A, I, E and ρ represented the respective cross-sectional area, area moment of inertia, Young modulus, and density; the subscripts o and L denoted the axial position of the given properties at $x = 0$ and $x = L$, respectively. The governing equation of motion of such a beam was expressed by a fourth-order partial differential equation as follows.[14,17]

$$\frac{\partial^2}{\partial x^2}\left[S(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] + m(x)\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad 0 \le x \le L. \tag{1}$$

Here w was the transverse displacement, L was the beam length, and x was the axial coordinate. $S(x) = E(x)I(x)$ was the bending rigidity where $E(x)$ and $I(x)$ were the Young modulus and area moment of inertia at position x, respectively. $m(x) = \rho(x)A(x)$ was the beam mass per unit length in which $\rho(x)$ and $A(x)$ were the respective density and cross-sectional area at location x. Assume that $w(x,t) = W(x)e^{i\omega t}$ and substitute it into Eq. (1) to yield:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[S(x)\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right] - m(x)\omega^2 W = 0, \quad 0 \le x \le L; \tag{2}$$

where ω was the natural frequency. Changing the spatial variables by $\xi = 2x/L - 1$ and letting $\Omega^2 = \omega^2 L^4/16$ allowed Eq. (2) to be rewritten as:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[S(\xi)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2}\right] - m(\xi)\Omega^2 W = 0, \quad -1 \le \xi \le 1; \tag{3}$$

or

$$S(\xi)\frac{\mathrm{d}^4 W}{\mathrm{d}\xi^4} + 2S'(\xi)\frac{\mathrm{d}^3 W}{\mathrm{d}\xi^3} + 2S''(\xi)\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} = m(\xi)\Omega^2 W, \quad -1 \le \xi \le 1. \tag{4}$$

The associated classic boundary conditions were represented as follows:

$$W = 0, \quad \frac{\mathrm{d}W}{\mathrm{d}\xi} = 0 \quad \text{(Clamped end)};$$
$$W = 0, \quad \frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} = 0 \quad \text{(Pinned end)};$$
$$\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\left[S(\xi)\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2}\right] = 0 \quad \text{(Free end)}. \tag{5}$$

The free bending vibration problem of non-uniform axially FG beams in Eqs. (4) and (5) were solved using the Chebyshev collocation method. The following Gauss-Chebyshev-Lobatto collocation points within the interval [-1, 1] were used:[35]

$$\xi_j = cos\frac{\pi j}{N}, \quad j = 0, 1, 2, \ldots, N. \tag{6}$$

Thus, the displacement function $W(\xi)$ was expanded by the Nth-order Chebyshev polynomials as follows:

$$W(\xi) \approx \sum_{N}^{j=0} \gamma_j(\xi) W(\xi_j), \quad j = 0, 1, 2, \ldots, N. \tag{7}$$

with

$$\gamma_j(\xi) = \frac{(-1)^{j+1}(1-\xi^2)T_N'(\xi)}{c_j N^2(\xi-\xi_j)};$$
$$T_N(\xi_j) = cos\left(N cos^{-1}(\xi_j)\right);$$
$$\gamma_j(\xi_k) = \delta_{jk};$$
$$c_j = \begin{cases} 2 & j=0, \quad N=0 \\ 1 & j=1,2,\dots, \quad N-1 \end{cases}. \tag{8}$$

Then, the first derivative of the displacement function $W(\xi)$ was obtained by the following matrix vector multiplication:

$$W'(\xi_j) = \sum_{j=0}^{N} (D_N)_{ij} W(\xi_j), \quad j=0,1,2,\dots,N. \tag{9}$$

Here $(D_N)_{ij} = \gamma_j'(\xi_j)$ was the i, j element of an $(N+1) \times (N+1)$ Chebyshev differentiation matrix D_N. The entries of this matrix were:[35]

$$(D_N)_{00} = \frac{2N^2+1}{6}; \quad (D_N)_{NN} = -\frac{2N^2+1}{6};$$
$$(D_N)_{jj} = -\frac{\xi_j}{2(1-\xi_j^2)}, \quad j=1,2,\dots,N-1;$$
$$(D_N)_{ij} = \frac{c_i(-1)^{i+j}}{c_j(\xi_i-\xi_j)}, \quad i \neq j, \quad j=1,2,\dots,N-1. \tag{10}$$

For simplicity, the first derivative of the Chebyshev differentiation matrix was denoted by D_1. The kth derivative was obtained by $D_k = (D_1)^k$.

Based on the Chebyshev collocation method as described, the ordinary differential equation in Eq. (4) was rewritten in terms of Chebyshev differentiation matrices as:

$$(K_1 D_4 + 2K_2 D_3 + K_3 D_2)\overline{W} = \Omega^2 M\overline{W}; \tag{11}$$

where

$$K_1 = \begin{bmatrix} S(\xi_0) & 0 & \dots & 0 \\ 0 & S(\xi_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S(\xi_N) \end{bmatrix};$$
$$K_2 = \begin{bmatrix} S'(\xi_0) & 0 & \dots & 0 \\ 0 & S'(\xi_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S'(\xi_N) \end{bmatrix};$$
$$K_3 = \begin{bmatrix} S''(\xi_0) & 0 & \dots & 0 \\ 0 & S''(\xi_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S''(\xi_N) \end{bmatrix};$$
$$m = \begin{bmatrix} m(\xi_0) & 0 & \dots & 0 \\ 0 & m(\xi_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m(\xi_N) \end{bmatrix};$$
$$\overline{W} = \{W(\xi_0)W(\xi_1)\dots W(\xi_N)\}^T = \{W_1 W_2 \dots W_{N+1}\}^T. \tag{12}$$

Likewise, the boundary equations in Eq. (5) at the supporting ends of the beam were also expressed by Chebyshev differentiation matrices as given in Table 1. When the homogeneous boundary conditions were imposed on the governing Eq. (11), some mathematical operations were performed. First, the first and second equations of the system were replaced by the two boundary conditions at the right end. The *N*th and (*N*+1)th equations were replaced by those at the left end. Then, the *N*th and (*N*+1)th equations of the new system were moved up to become the third and fourth equations. Finally, by shifting the displacements W_N and W_{N+1} to the third and fourth rows of the displacement vector $\overline{W}$, the generalized eigenvalue problem was obtained as:

$$\begin{bmatrix} K_{BB} & K_{BI} \\ K_{IB} & K_{II} \end{bmatrix} \begin{Bmatrix} \overline{W_B} \\ \overline{W_I} \end{Bmatrix} = \Omega^2 \begin{bmatrix} O & O \\ M_{IB} & M_{II} \end{bmatrix} \begin{Bmatrix} \overline{W_B} \\ \overline{W_I} \end{Bmatrix}; \tag{13}$$

$$\overline{W_B} = \{W_1 \quad W_2 \quad W_N \quad W_{N+1}\}^T;$$
$$\overline{W_I} = \{W_3 \quad W_4 \quad \dots \quad W_{N-1}\}^T. \tag{14}$$

The subscripts B and I denoted the boundary and internal collocation points associated with the boundary condition and the governing equation, respectively. The sizes of the stiffness matrices K_{BB}, K_{BI}, K_{IB} and K_{II} were 4×4, $4 \times (N-3)$, $(N-3) \times 4$ and $(N-3) \times (N-3)$, respectively. The sizes of the inertia matrices M_{IB} and M_{II} were the same as the corresponding stiffness matrices K_{IB} and K_{II}.

To solve the general eigenvalue equation in Eq. (13), it was expanded to yield:

$$K_{BB}\overline{W_B} + K_{BI}\overline{W_I} = 0; \tag{15}$$
$$K_{IB}\overline{W_B} + K_{II}\overline{W_I} = \Omega^2\left(M_{IB}\overline{W_B} + M_{II}\overline{W_I}\right). \tag{16}$$

Then, after introducing Eq. (15) into Eq. (16), the algebraic eigenvalue equation was reduced to the following form:

$$A\overline{W_I} = \Omega^2 B\overline{W_I}; \tag{17}$$

$$A = -K_{IB}(K_{BB})^{-1}K_{BI} + K_{II};$$
$$B = -M_{IB}(K_{BB})^{-1}K_{BI} + M_{II}. \tag{18}$$

Thus, natural frequencies of the free bending vibration of non-uniform axially FG Euler-Bernoulli beams with various classical boundary conditions can be obtained as the solution:

$$det\left(A - \Omega^2 B\right) = 0. \tag{19}$$

3. RESULTS AND DISCUSSIONS

To assure the successful application of the Chebyshev collocation method in the vibration analysis of axially FG Euler-Bernoulli beams with non-uniform cross-sections, the accuracy studies are carried out through various numerical examples. First example to be concerned is a uniform axially FG beam with the bending rigidity $S(x)$ and mass per unit length $m(x)$ according to the following distribution:[14]

$$S(x) = \left[E_o + (E_L - E_o)\frac{e^{\alpha x/L}-1}{e^\alpha - 1},\right] I \quad \alpha \neq 0;$$
$$m(x) = \left[\rho_o + (\rho_L - \rho_o)\frac{e^{\alpha x/L}-1}{e^\alpha - 1}\right] A, \quad \alpha \neq 0; \tag{20}$$

Table 1. Boundary condition equations in terms of Chebyshev differentiation matrices.

Boundary Conditions	Left end ($\xi = -1$)	Right end ($\xi = +1$)
clamped	$[0 \quad 0 \quad \cdots \quad 1]\,\overline{W} = 0$ $[D_1(N+1,:)]\,\overline{W} = 0$	$[1 \quad 0 \quad \cdots \quad 0]\,\overline{W} = 0$ $[D_1(1,:)]\,\overline{W} = 0$
pinned	$[0 \quad 0 \quad \cdots \quad 1]\,\overline{W} = 0$ $[D_2(N+1,:)]\,\overline{W} = 0$	$[1 \quad 0 \quad \cdots \quad 0]\,\overline{W} = 0$ $[D_2(1,:)]\,\overline{W} = 0$
free	$[D_2(N+1,:)]\,\overline{W} = 0$ $S(\xi_N)\,[D_3(N+1,:)]\,\overline{W} + S(\xi_N)\,[D_2(N+1,:)]\,\overline{W} = 0$	$[D_2(1,:)]\,\overline{W} = 0$ $S(\xi_0)\,[D_3(1,:)]\,\overline{W} + S(\xi_0)\,[D_2(1,:)]\,\overline{W} = 0$

Table 2. Comparison of dimensionless fundamental frequencies of axially FG beams for different values of α.

α	source	CF	PP	CP	CC
-10	Present solution	3.5158	11.4560	16.4022	24.0528
	Huang[14]	3.5656	11.4532	16.4775	24.0576
	Hein[16]	NA	11.4481	16.3837	24.0269
	Liu[23]	3.5340	11.4560	16.4111	24.0660
-3	Present solution	3.1409	11.2443	16.0255	23.9433
	Huang[14]	3.1421	11.2443	16.0219	23.9456
	Hein[16]	NA	11.2422	16.0307	23.9384
	Liu[23]	3.1410	11.2443	16.0262	23.9446
3	Present solution	2.8545	10.3669	15.7169	24.9364
	Huang[14]	2.8544	10.3669	15.7171	24.9375
	Hein[16]	NA	10.3670	15.7171	24.9371
	Liu[23]	2.8545	10.3669	15.7169	24.9366
10	Present solution	3.0431	9.9366	15.4957	24.8063
	Huang[14]	3.0431	9.9358	15.4956	24.7949
	Hein[16]	NA	9.9366	15.4930	24.8080
	Liu[23]	3.0431	9.9366	15.4958	24.8068

where α is the material graded index which describes the distribution of material properties. For $\alpha > 0$, a larger value α (e.g., 10) represents a more sudden increase in the properties $E(x)/E_o$ and $\rho(x)/\rho_o$ near the right surface. In contrast, the properties vary abruptly near the left surface for a smaller value of α (e.g., -10) as $\alpha < 0$. The beam is made of aluminum and zirconia, whose properties vary axially according to the exponential function in Eq. (20). Meanwhile, the beam is aluminum-rich at $x = 0$ and zirconia-rich at $x = L$. The dimensionless fundamental natural frequencies $\lambda = \omega L^2 \left(\rho_o A / E_o I\right)^{1/2}$ of the beams with various values of α and boundary conditions are presented in Table 2. The present results agree well with those given by Huang and Li, Hein and Feklistova and Liu et al.[14,16,23] The results reveal that the restraint and material graded index have a significant impact on the frequencies. The fundamental frequency decreases first and increases with the increasing α for the clamped-free (CF) beam, reduces with the increasing α for the pinned-pinned (PP) and clamped-pinned (CP) beam, and varies irregularly with α for the clamped-clamped (CC) beam.

Secondly, a non-uniform axially FG beam is examined whose bending rigidity $S(x)$ and mass per unit length $m(x)$ are represented as:[17]

$$S(x) = E(x)I(x) = \left[E_o\left(1+\frac{x}{L}\right)\right]\left[\left(1-C_b\frac{x}{L}\right)\left(1-C_h\frac{x}{L}\right)^3 I_o\right];$$
$$m(x) = \rho(x)A(x) = \left[\rho_o\left(1+\frac{x}{L}+\left(\frac{x}{L}\right)^2\right)\right]\left[\left(1-C_b\frac{x}{L}\right)\left(1-C_h\frac{x}{L}\right)A_o\right]. \tag{21}$$

Here C_b and C_h denote the width and height taper ratios, respectively, whose values range from 0 to 1. The non-uniform beam becomes prismatic as $C_b = C_h = 0$, and it tapers to a point at $x = L$ as $C_b = C_h = 1$. The tapered FG beam has the properties of $L = 1$ m, $A_o = 4 \times 10^{-4}$ m^2, $I_o = 1.33 \times 10^{-8}$ m^4, $E_o = 70$ Gpa and $\rho = 2702$ kgm^{-3}. Its material properties vary axially according to the polynomial function in Eq. (21). The dimensionless frequency parameter $\lambda = \omega L^2 \left(\rho_o A_o / E_o I_o\right)^{1/2}$ is used in the calculations of natural frequencies. Tables 3– 5 present the variations of the dimensionless fundamental frequencies against different taper ratios for axially FG beams with CF, PP and CC boundary conditions, respectively. In comparison with the results obtained by Shahba and Rajasekaran and Liu et al., an excellent agreement is achieved.[17,23] Depending on the taper ratios and boundary conditions, the increase in height and width taper ratios may decrease or increase the fundamental natural frequencies. For the beams with the same width ratio C_b, the natural frequencies reduce with the increasing height taper ratio C_h except for those of CF beams. The frequencies of CF beams enlarge with the increasing C_h. For the beams with the same height taper ratio C_h, the frequencies of CF beams increase with the increasing width taper ratio C_b but those of PP and CC beams vary differently with C_b depending on the value of C_h. It is important to note that the height taper ratio has a more profound impact on the natural frequencies of all beams than width taper ratio while it shows an opposite trend for CF beams.

From the previous comparison study, it indicates that the proposed method can be applied to evaluate the free vibration frequencies of various axially FG tapered beams with high accuracy. In the next, the free vibration of non-uniform beams with axially graded material properties according to a power-law function is studied to show the adaptability of the presented method. Its bending rigidity $S(x)$ and mass per unit length $m(x)$ are given as follows:[17,19]

$$S(x) = \left[E_o + (E_L - E_o)\left(\frac{x}{L}\right)^p\right]\left[\left(1-C_b\frac{x}{L}\right)\left(1-C_h\frac{x}{L}\right)^3 I_o\right];$$
$$m(x) = \left[\rho_o + (\rho_L - \rho_o)\left(\frac{x}{L}\right)^p\right]\left[\left(1-C_b\frac{x}{L}\right)\left(1-C_h\frac{x}{L}\right)A_o\right]. \tag{22}$$

Here the non-negative exponent p is the volume fraction index. For a larger value of p, the properties $E(x)/E_o$ and $\rho(x)/\rho_o$ change more suddenly near the right surface and the material at the left surface is the dominant constituent. For a smaller value of p, the variation of the properties shows an opposite tendency. The recommended value of p ranges from 1/3 to 3 to insure that the FG material has a proper balance between the percentages of the constituents.[15] The axially FG beam is composed of zirconia and aluminum with ceramic-rich left side and metal-rich right side. The properties of the beam are

Table 3. Comparison of dimensionless fundamental frequencies of CF axially FG beams with different tapered ratios.

C_h	source	$C_b = 0$	$C_b = 0.2$	$C_b = 0.4$	$C_b = 0.6$	$C_b = 0.8$
0	Present solution	2.4256	2.6054	2.8508	3.2137	3.8310
	Shahba[17]	2.4256	2.6054	2.8508	3.2137	3.8310
	Liu[23]	2.4256	2.6054	2.8508	3.2137	3.8310
0.2	Present solution	2.5051	2.6863	2.9336	3.2994	3.9219
	Shahba[17]	2.5051	2.6863	2.9336	3.2994	3.9220
	Liu[23]	2.5051	2.6863	2.9336	3.2993	3.9219
0.4	Present solution	2.6155	2.7987	3.0486	3.4181	4.0471
	Shahba[17]	2.6155	2.7988	3.0486	3.4181	4.0471
	Liu[23]	2.6155	2.7987	3.0486	3.4181	4.0471
0.6	Present solution	2.7835	2.9699	3.2237	3.5985	4.2355
	Shahba[17]	2.7836	2.9699	3.2237	3.5985	4.2355
	Liu[23]	2.7836	2.9699	3.2237	3.5985	4.2355
0.8	Present solution	3.0871	3.2794	3.5401	3.9232	4.5695
	Shahba[17]	3.0871	3.2794	3.5401	3.9232	4.5695
	Liu[23]	3.0871	3.2794	3.5401	3.9232	4.5695

Table 4. Comparison of dimensionless fundamental frequencies of PP axially FG beams with different tapered ratios.

C_h	source	$C_b = 0$	$C_b = 0.2$	$C_b = 0.4$	$C_b = 0.6$	$C_b = 0.8$
0	Present solution	9.0286	9.0599	9.0867	9.0994	9.0685
	Shahba[17]	9.0285	9.0599	9.0867	9.0994	9.0685
	Liu[23]	9.0286	9.0599	9.0867	9.0994	9.0685
0.2	Present solution	8.1341	8.1462	8.1498	8.1336	8.0645
	Shahba[17]	8.1341	8.1462	8.1498	8.1336	8.0646
	Liu[23]	8.1341	8.1462	8.1498	8.1336	8.0646
0.4	Present solution	7.1530	7.1455	7.1254	7.0794	6.9703
	Shahba[17]	7.1531	7.1455	7.1254	7.0794	6.9703
	Liu[23]	7.1531	7.1455	7.1254	7.0794	6.9703
0.6	Present solution	6.0357	6.0082	5.9637	5.8868	5.7351
	Shahba[17]	6.0357	6.0082	5.9637	5.8868	5.7351
	Liu[23]	6.0357	6.0082	5.9638	5.8868	3.7351
0.8	Present solution	4.6519	4.6045	4.5353	4.4263	4.2281
	Shahba[17]	4.6520	4.6045	4.5354	4.4263	4.2281
	Liu[23]	4.6520	4.6046	4.5355	4.4264	4.2284

Table 5. Comparison of dimensionless fundamental frequencies of CC axially FG beams with different tapered ratios.

C_h	source	$C_b = 0$	$C_b = 0.2$	$C_b = 0.4$	$C_b = 0.6$	$C_b = 0.8$
0	Present solution	20.4721	20.4151	20.2883	20.0186	19.3844
	Shahba[17]	20.4721	20.4151	20.2883	20.0186	19.3844
	Liu[23]	20.4721	20.4152	20.2883	20.0186	19.3845
0.2	Present solution	18.2170	18.1995	18.1286	17.9436	17.4564
	Shahba[17]	18.2170	18.1995	18.1286	17.9436	17.4564
	Liu[23]	18.2170	18.1996	18.1286	17.9736	17.4565
0.4	Present solution	15.8281	15.8497	15.8349	15.7366	15.4021
	Shahba[17]	15.8281	15.8497	15.8349	15.7366	15.4021
	Liu[23]	15.8282	15.8498	15.8350	15.7367	15.4025
0.6	Present solution	13.2291	13.2894	13.3316	13.3234	13.1521
	Shahba[17]	13.2291	13.2894	13.3316	13.3234	13.1521
	Liu[23]	13.2294	13.2896	13.3319	13.3238	13.1529
0.8	Present solution	10.2217	10.3211	10.4234	10.5143	10.5301
	Shahba[17]	10.2217	10.3211	10.4234	10.5143	10.5301
	Liu[23]	10.2236	10.3231	10.4257	10.5170	10.5343

as follows: $E_o = 200$ Gpa, $\rho_o = 5700$ kgm^{-3}, $E_L = 70$ Gpa, $\rho_L = 2702$ kgm^{-3}, and $r^2 = I_o/A_oL^2 = 0.0001$. r is the inverse slenderness ratio. The effects of the taper ratios, material graded indices and restraint types on the vibration frequencies of such axially FG beams are demonstrated. The frequency parameter $\lambda = \omega L^2 \left(\rho_o A_o/E_o I_o\right)^{1/2}$ is used to evaluate the dimensionless natural frequencies.

Tables 6 and 7 present the effects of various values of C_h and C_b on the first three dimensionless frequencies for the CF and PP axially FG beam with $p = 2$, respectively. The variations of the first three natural frequencies with respect to the taper ratios C_h and C_b for the CP and CC axially FG beams are depicted in Figs. 2 and 3 to show the varying trend of the natural frequencies. As seen in Table 6, the first frequency of the CF beam dramatically increases with the increase of taper ratios C_h and C_b. However, its second and third frequencies reduce with C_h but increase with C_b. It is also noted that the first frequency is affected much more by the increasing width taper ratio C_b than the height taper ratio C_h. In contrast, the height taper ratio has a more profound influence on the higher mode frequencies of the CF beams. As shown in Table 7, the increase in height taper ratio C_h has a significant impact on the

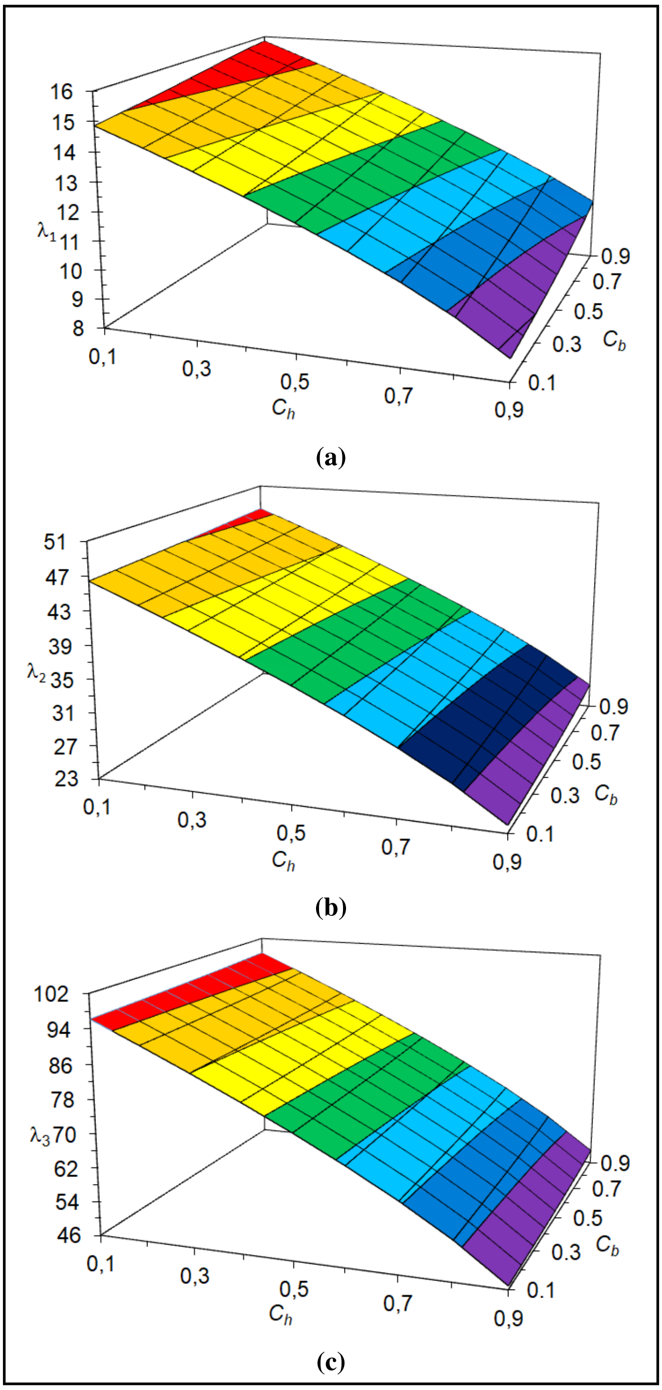

Figure 2. Variation of dimensionless natural frequencies against taper ratios for CP axially FG beams with $p = 2$. (a) First mode (b) Second mode (c) Third mode.

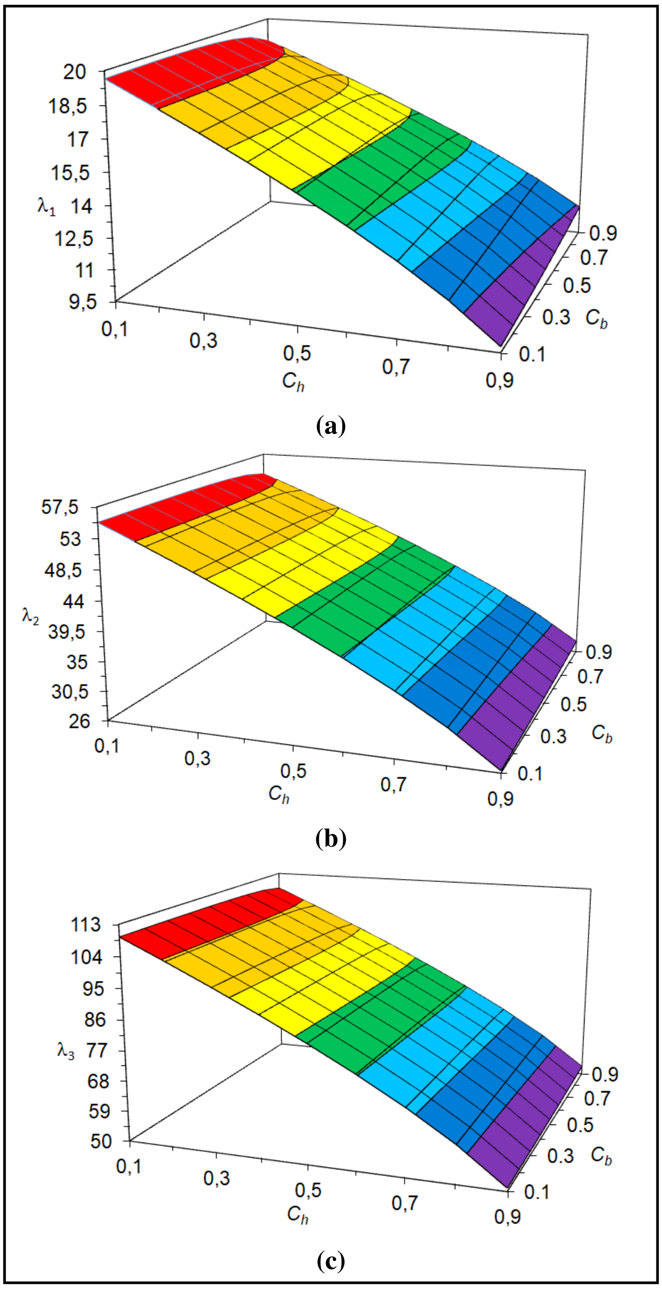

Figure 3. Variation of dimensionless natural frequencies against taper ratios for CC axially FG beams with $p = 2$. (a) First mode (b) Second mode (c) Third mode.

reduction of the first three frequencies of the PP beams. The first frequency is most significantly affected by the increasing C_h, followed by the second and third frequencies. When the width taper ratio C_b increases, the first frequency slightly reduces but the second and third frequencies increase. The only exception is the second frequency of the beam with $C_h = 0.1$, which is decreased as C_b is increased from 0.8 to 0.9. As observed, the height taper ratio will enhance the effect of width taper ratio on the natural frequencies, especially for the first mode. It can be found in Fig. 2 that all three frequencies of CP beams reduce with the increase in C_h and slightly increase with C_b. Unlike the PP beams, the increasing C_h has a more profound effect on the reduction rate of higher mode frequencies. Like the PP and CP beams, Fig. 3 reveals that the first three frequencies of CC beams decrease as the height taper ratio C_h increases. Like the CP beams, the reduction rate of higher mode frequency with respect to the increasing height taper ratio is higher than that of lower ones. With the increase in C_b, the first three frequencies of CC beams slightly increase first and then reduce. However, the effect of the increasing C_b on the reduction of frequencies is gradually diminished while the CC beam has a higher height taper ratio C_h.

To summarize the results reported previously, several conclusions on the effects of taper ratios for axially FG beams with various boundary conditions can be made as follows. In general, the height taper ratio remarkably affects the natural frequencies of all beams than width taper ratio while it shows an opposite tendency for the fundamental frequencies of CF beams. With the increase in height taper ratio C_h, the natural frequencies decrease for all beams with the same width ratio

Table 6. Dimensionless frequencies λ_i of CF axially FG Z/A beams with different tapered ratios ($p = 2$).

C_b	mode	$C_h = 0.1$	$C_h = 0.2$	$C_h = 0.3$	$C_h = 0.4$	$C_h = 0.5$	$C_h = 0.6$	$C_h = 0.7$	$C_h = 0.8$	$C_h = 0.9$
0.1	λ_1	4.4645	4.5164	4.5773	4.6501	4.7394	4.8524	5.0019	5.2135	5.5488
	λ_2	22.6987	21.9220	21.1204	20.2904	19.4283	18.5303	17.5950	16.6337	15.7326
	λ_3	59.3274	56.5197	53.6280	50.6359	47.5210	44.2511	40.7785	37.0286	32.9061
0.2	λ_1	4.6167	4.6683	4.7288	4.8013	4.8903	5.0029	5.1521	5.3637	5.6998
	λ_2	22.9514	22.1659	21.3552	20.5159	19.6441	18.7360	17.7901	16.8174	15.9038
	λ_3	59.5887	56.7744	53.8757	50.8761	47.7530	44.4740	40.9912	37.2294	33.0918
0.3	λ_1	4.7926	4.8438	4.9040	4.9762	5.0647	5.1769	5.3258	5.5372	5.8737
	λ_2	23.2426	22.4469	21.6258	20.7758	19.8930	18.9735	18.0155	17.0299	16.1022
	λ_3	59.8934	57.0710	54.1636	51.1547	48.0216	44.7317	41.2367	37.4608	33.3060
0.4	λ_1	4.9990	5.0499	5.1098	5.1815	5.2695	5.3813	5.5297	5.7407	6.0772
	λ_2	23.5853	22.7778	21.9446	21.0821	20.1864	19.2535	18.2815	17.2809	16.3372
	λ_3	60.2581	57.4251	54.5067	51.4860	48.3403	45.0369	41.5269	37.7340	33.5586
0.5	λ_1	5.2460	5.2966	5.3560	5.4273	5.5149	5.6261	5.7738	5.9841	6.3200
	λ_2	24.0004	23.1788	22.3311	21.4537	20.5426	19.5937	18.6049	17.5866	16.6240
	λ_3	60.7101	57.8630	54.9298	51.8937	48.7315	45.4105	41.8813	38.0669	33.8664
0.6	λ_1	5.5487	5.5989	5.6580	5.7289	5.8160	5.9266	6.0736	6.2828	6.6168
	λ_2	24.5237	23.6846	22.8189	21.9231	20.9929	20.0242	19.0147	17.9746	16.9890
	λ_3	61.2980	58.4310	55.4772	52.4196	49.2348	45.8896	42.3344	38.4914	34.2583
0.7	λ_1	5.9315	5.9816	6.0405	6.1112	6.1979	6.3079	6.4539	6.6613	6.9914
	λ_2	25.2224	24.3607	23.4718	22.5521	21.5972	20.6028	19.5666	18.4985	17.4837
	λ_3	62.1188	59.2221	56.2375	53.1477	49.9294	46.5488	42.9557	39.0717	34.7933
0.8	λ_1	6.4368	6.4871	6.5463	6.6170	6.7037	6.8133	6.9581	7.1627	7.4852
	λ_2	26.2411	25.3485	24.4279	23.4754	22.4867	21.4573	20.3846	19.2785	18.2240
	λ_3	63.4024	60.4565	57.4211	54.2786	51.0053	47.5669	43.9127	39.9631	35.6144
0.9	λ_1	7.1449	7.1965	7.2569	7.3287	7.4162	7.5259	7.6694	7.8689	8.1755
	λ_2	27.9572	27.0195	26.0526	25.0526	24.0149	22.9350	21.8104	20.6506	19.5378
	λ_3	65.8665	62.8267	59.6945	56.4517	53.0740	49.5264	45.7570	41.6856	37.2102

C_b except for the fundamental frequencies of CF beams. The fundamental frequencies of CF beams show an ascendant tendency with the increasing C_h. When the width taper ratio C_b increases, the first three frequencies increase for CF and CP beams with the same height taper ratio C_h, but those of PP and CC beams vary differently depending on the value of C_h. Therefore, it is difficult to predict the natural frequencies for axially FG tapered beams of various boundary conditions with respect to the taper ratios.

Table 8 presents the effects of material gradation on the first three frequencies of axially FG beams with $C_h = 0.3$ and $C_b = 0$ under different boundary conditions. As can be seen, the variations of natural frequencies of axially FG tapered beams with respect to p depend on the boundary conditions. With the increasing p, the first and second modes of CF beam increase first, and then decrease while the third mode continues to increase. All modes of PP beam enlarge with p. For CP and CC beams, the fundamental frequency increases and decreases alternatively with p, while other two modes show an increasing trend. As cited by Shahba et al., it is important to note that the effects of p on the variation of the natural frequencies of tapered axially FG beams are hard to be predicted because both the stiffness and mass of the beam are enhanced with the increase in p.[15]

4. CONCLUSIONS

The bending vibration of various axially FG Euler-Bernoulli beams with tapered cross section is studied based on the Chebyshev collocation method. The effects of the material and cross-sectional properties varying along the beam length direction on the vibration behaviors are investigated. Natural frequencies for the uniform axially FG beams with the exponential function gradient and the tapered axially FG beams with polynomial function gradient are evaluated and compared with the published ones to confirm the effectiveness of the present method. The results indicate that the present study can analyse the free vibration of Euler-Bernoulli beams with different axially graded material properties and varying cross-sectional properties under various boundary conditions. Finally, the axially FG tapered beam with power law gradient are examined to demonstrate the adaptability of the present method to different graded material properties. Hence, it is believed that the present method can be extended to study the dynamic problem of elastically supported bi-directional FG beams resting on elastic foundations in the future work.

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Table 7. Dimensionless frequencies λ_i of PP axially FG Z/A beams with different tapered ratios ($p = 2$).

C_b	mode	$C_h = 0.1$	$C_h = 0.2$	$C_h = 0.3$	$C_h = 0.4$	$C_h = 0.5$	$C_h = 0.6$	$C_h = 0.7$	$C_h = 0.8$	$C_h = 0.9$
0.1	λ_1	9.0338	8.4882	7.9192	7.3217	6.6881	6.0071	5.2595	4.4079	3.3536
	λ_2	36.3792	34.4110	32.3917	30.3101	28.1503	25.8878	23.4819	20.8544	17.8076
	λ_3	81.7246	77.2679	72.6821	67.9385	62.9962	57.7930	52.2253	46.0948	38.9010
0.2	λ_1	9.0096	8.4606	7.8882	7.2875	6.6508	5.9671	5.2171	4.3639	3.3099
	λ_2	36.3993	34.4352	32.4202	30.3430	28.1878	25.9303	23.5300	20.9091	17.8717
	λ_3	81.7592	77.3082	72.7281	67.9902	63.0536	57.8562	52.2946	46.1703	38.9835
0.3	λ_1	8.9793	8.4265	7.8506	7.2463	6.6064	5.9196	5.1672	4.3123	3.2586
	λ_2	36.4238	34.4644	32.4543	30.3822	28.2323	25.9806	23.5868	20.9739	17.9481
	λ_3	81.8024	77.3577	72.7839	68.0523	63.1223	57.9315	52.3767	46.2599	39.0816
0.4	λ_1	8.9407	8.3839	7.8041	7.1961	6.5526	5.8626	5.1075	4.2510	3.1977
	λ_2	36.4535	34.4997	32.4952	30.4290	28.2855	26.0407	23.6549	21.0518	18.0408
	λ_3	81.8568	77.4191	72.8525	68.1282	63.2055	58.0224	52.4758	46.3678	39.2002
0.5	λ_1	8.8910	8.3298	7.7457	7.1336	6.4863	5.7928	5.0347	4.1765	3.1240
	λ_2	36.4895	34.5422	32.5444	30.4853	28.3495	26.1132	23.7374	21.1469	18.1551
	λ_3	81.9260	77.4964	72.9379	68.2220	63.3081	58.1341	52.5973	46.5005	39.3467
0.6	λ_1	8.8253	8.2594	7.6706	7.0540	6.4024	5.7051	4.9439	4.0838	3.0323
	λ_2	36.5321	34.5926	32.6031	30.5530	28.4268	26.2015	23.8386	21.2648	18.2992
	λ_3	82.0148	77.5947	73.0460	68.3402	63.4368	58.2742	52.7498	46.6676	39.5327
0.7	λ_1	8.7360	8.1647	7.5708	6.9491	6.2928	5.5912	4.8266	3.9645	2.9141
	λ_2	36.5792	34.6498	32.6711	30.6326	28.5193	26.3087	23.9636	21.4132	18.4855
	λ_3	82.1288	77.7205	73.1841	68.4912	63.6015	58.4538	52.9463	46.8846	39.7778
0.8	λ_1	8.6082	8.0309	7.4311	6.8038	6.1421	5.4356	4.6669	3.8028	2.7527
	λ_2	36.6200	34.7046	32.7408	30.7187	28.6238	26.4346	24.1158	21.6009	18.7330
	λ_3	82.2700	77.8780	73.3586	68.6839	63.8140	58.6883	53.2065	47.1774	40.1165
0.9	λ_1	8.4087	7.8245	7.2176	6.5835	5.9152	5.2025	4.4286	3.5610	2.4988
	λ_2	36.6051	34.7113	32.7710	30.7748	28.7094	26.5550	24.2799	21.8256	19.0716
	λ_3	82.4011	78.0360	73.5454	68.9019	64.0666	58.9805	53.5464	47.5800	40.6019

Table 8. Dimensionless frequencies λ_i of axially FG Z/A beams with different material graded indices and boundary conditions ($C_h = 0.3, C_b = 0$).

BC	mode	$p = 0.2$	$p = 0.5$	$p = 1$	$p = 2$	$p = 3$	$p = 5$	$p = 10$
CF	λ_1	3.8182	4.2269	4.4481	4.4445	4.3455	4.1802	3.9757
	λ_2	18.8631	19.8555	20.4650	20.9133	21.1089	21.2124	21.0340
	λ_3	49.1905	50.9556	52.1801	53.4105	54.1516	54.9396	55.3746
PP	λ_1	7.3610	7.5499	7.7361	7.9451	8.0613	8.1781	8.2655
	λ_2	29.9177	30.8992	31.7175	32.3676	32.6260	32.8720	33.1385
	λ_3	67.2914	69.4157	71.1670	72.6438	73.2911	73.8883	74.4378
CP	λ_1	12.9351	13.4722	13.6218	13.5547	13.5187	13.5311	13.5921
	λ_2	39.4242	40.6473	41.2558	41.6533	41.8721	42.1463	42.4794
	λ_3	80.8015	83.1292	84.5982	85.8777	86.5591	87.2573	87.9037
CC	λ_1	17.5197	17.9293	17.8606	17.5584	17.4236	17.3770	17.5106
	λ_2	47.6082	48.7865	49.1963	49.2651	49.2672	49.2927	49.4399
	λ_3	92.7090	95.0888	96.4150	97.3340	97.7103	97.9841	98.1706

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