Determination of Notched Input for Equipment Under Random Vibrations

K. Renji

Advanced Technology Development Group, UR Rao Satellite center, ISRO Vimanapura Post, Bangalore, India 560017. E-mail: renjikna@gmail.com

(Received 22 June 2021; accepted 4 November 2021)

Equipment that is mounted on a spacecraft is subjected to random vibration tests to verify whether they can withstand the specified random loads. These tests are generally carried out by using shaker systems during which equipment experiences very high responses at the natural frequencies of the equipment. To reduce such overtesting, notching of the input is done. Notching of the input is normally carried out by considering the force generated at the base and limiting it to a specified value. To accomplish the notching, the force spectrum to be limited and measurement of base force during the tests are needed. This work shows that the acceleration input at the interface of equipment gets reduced at its resonance frequency and this feature can be utilized in arriving at the notched input. An expression to determine the depth of notching is derived and the results are compared with those obtained using numerical simulations. The depth of the notch increases with the response of the oscillator and it is sensitive to the stiffness ratios rather than the mass ratios of the oscillator and the mounting panel. This behavior and the expressions derived can be effectively used in arriving at the notched input for an equipment without the need for measuring the base force, especially for random vibration testing, which is demonstrated with an example.

NOMENCLATURE

$c_1,\ c_2$	Damping coefficients of the 2DOF system
f	Frequency of excitation in Hz
$f_1,\ f_2$	Natural frequencies of the 2DOF system, in Hz
$f_{11},\ f_{22}$	Natural frequencies of the 2 oscillators of the 2DOF system, in Hz
$f_1\,(t)\,, f_2(t)$	Forcing function at DOF1 and DOF2
$\overline{f_1(t)},\ \overline{f_2(t)}$	Generalised forces at DOF1 and DOF2
$F_1,\ F_2$	Amplitudes of the forces at DOF1 and DOF2
j	Complex operator
$k_1,\ k_2$	Stiffness values of the 2DOF system
$m_1,\ m_2$	Masses of the 2DOF system
$q_1,\ q_2$	Generalised coordinates of the 2DOF system
Q_1, Q_2	Quality factors of the 2DOF system
Q_{11}, Q_{22}	Quality factors (amplifications) of the two SDOF systems that constitute the 2DOF system
${u_1}^i,\ {u_2}^i$	Modal vectors of ith mode
$x_1,\ x_2$	Displacements of degree-of freedom 1 and 2
$X_1,\ X_2$	Amplitudes of the displacements of degree-of freedom 1 and 2
${X_1}^2,\ {X_2}^2$	Mean square values of displacements
δ	Depth of notching
ω	Frequency of excitation, in rad/s
$\omega_1,\ \omega_2$	Natural frequencies of the 2DOF system, in rad/s
$\omega_{11},\ \omega_{22}$	Natural frequencies of the 2 oscillators of the 2DOF system, in rad/s
$\zeta_1,\ \zeta_2$	Modal damping factors of the 2DOF system
$\zeta_{11},\ \zeta_{22}$	Damping factors of the two SDOF systems that constitute the 2DOF system

1. INTRODUCTION

Equipment mounted in a spacecraft experiences random vibrations through their mounting platforms, called equipment panels. Equipment needs to be designed and tested to withstand these random loads. Usually, they are specified as base random accelerations defined in terms of Power Spectral Density (PSD) of the acceleration, expressed in terms of g^2/Hz. A typical random vibration specification of a spacecraft's equipment is such that the PSD of acceleration is 0.2 g^2/Hz for a frequency band of 100 to 700 Hz. Usually, the PSD of acceleration is reduced from 700 Hz to 2000 at a rate of 6 dB/oct.

The random vibration loads of the equipment are derived from the accelerations experienced at their interfaces with the mounting platform.[1] These accelerations are determined by being measured while an acoustic test is performed on the spacecraft. As equipment needs to be compatible across many spacecrafts, dependence on the measured responses and spacecraft details for arriving at the random loads cause several difficulties for the equipment manufacturer. Specifying an envelope of all possible levels as the specifications is not a prudent solution as the levels will be very high. Girard and Moreau[2] developed methodologies to arrive at the random vibration load as a function of the mass of the equipment, i.e., independent of spacecraft, which is a great step forward.

In the configuration of the system for the acoustic test, the pieces of equipment are mounted on the corresponding platform or equipment panel. However, while performing the random vibration testing of the equipment, the equipment is directly mounted on to the shaker system, referred here as hard mounted. The panels are absent during these tests. The responses of the equipment when they are tested under hard mounted conditions are significantly higher compared to the responses experienced when they are mounted on panels, even if they experience the same accelerations at their interfaces. As pointed out by Saltor[3] the random vibration loads specified should not be a mere translation of such measured numbers.

Strong resonances are exhibited when they are tested in hard mounted conditions that are not that significant when they are mounted on the panels. To reduce the severity during the hard mounted conditions, Girard and Moreau[1] employed concepts like Random Response Spectra (RRS) in arriving at the random vibration loads. Another practice is to reduce the input at the natural frequencies of the equipment, called notching of the input, such that the responses of the equipment when tested at hard mounted conditions are not significantly high compared to the responses expected if they were mounted on the panels.

The notched input can be arrived at using several methods. One method is to look for the acceleration measured at the interface of the equipment with the panel at the natural frequency of the hard mounted equipment. This method is generally used in random vibration testing. The other method is to use the responses measured on the equipment, which is employed in arriving at the notching during sine vibration testing. In sine vibration, these responses can be determined based on a rigorous structural analysis using a mathematical model where one relies on the measured accelerations in the case of random vibrations. As this method is based on the measured results, it becomes applicable only for either identical or similar spacecraft. Therefore, if the equipment is mounted on a different spacecraft, one needs to realize the spacecraft as well as perform the acoustic testing and measure the responses. This is a difficult sequence. Additionally, the equipment manufacturer must be dependent on the spacecraft details for the testing of their equipment. Determination of the notched input independent of these tests is the most desired. Though the work by Girard and Moreau[2] provides the framework for the random vibration specification of the equipment such that that they are independent of the spacecraft, it does not address the notching.

Several works have been carried out for arriving at suitable notching of the input, based on various considerations. Some of the earlier works are by Scharton,[4] Girard et. al.[5] and Girard and Newerla.[6] Among them, the most used methodology is force limited testing. Force limited testing is generally employed in random vibration testing. In this method, the base force developed during the random vibration testing is measured and limited to the base force expected when it is mounted on the spacecraft. The challenge now is arriving at the base force to be limited. Scharton[4] developed a methodology to arrive at the base force to be limited and presented a monograph for the force limited vibration testing of equipment. This methodology is now well established and followed by many spacecraft industries.[7,8] This technique is further extended even to spacecraft testing. A satellite can be considered to be equipment mounted on the vehicle structure. Methodology was demonstrated for the testing of the CASSINI spacecraft[9,10] and the same methodology, with some variations, is being used even today.[11–14] By adopting the above methodologies, over-testing of the equipment is reduced to a large extent.

The major difficulty in employing the force limited vibration testing is the need for measuring the base force during the vibration testing. This calls for special arrangements, force sensors and processing. Another issue is in arriving at the base force to be limited. In simple terms, the base force to be limited shall be the force that is expected at the base of the equipment when the equipment is mounted on to the spacecraft. This can be specified as a frequency spectrum, specifically PSD in the case of random vibrations. Determining the force expected at the base of the equipment when mounted on the equipment panel makes the process of force limiting complex. One should have the theoretical model or measured values and the PSD of force needs to be specified to the equipment manufacturer as well. In the absence of this information, many manufacturers limit peak value (in time domain) of the force at the base of the equipment to the quasi-static load specified. In many reported works,[4,7] the PSD of the force to be limited is worked out by multiplying the PSD of acceleration at the interface of the equipment by the mass of the equipment. Though it provides a simple way for employing force limited vibration testing, the above methodology of defining the PSD of the force at the base does not consider other parameters and it is approximate. However, if the notching of the acceleration can directly be defined, it will be very convenient as the measurement of force can be avoided. It also helps the designer in having the information on the expected input acceleration at the resonance during the design stage itself.

The objective of this work is to explore the possibility of whether a unique value of acceleration can be defined at the resonance of the equipment and if so to derive an expression to arrive at this acceleration. In this work, phenomenon of the natural reduction of the input (acceleration) at the natural frequency of the equipment is shown analytically when equipment is mounted on a platform and is absent when tested under hard mounted conditions. This behavior is conveniently used in arriving at the notched input. Subsequently an expression to estimate the dip in the acceleration input is derived which is a function of various structural parameters of the system. The methodology does not predict the exact input required, instead it provides the notched input as a factor of the un-notched input. Using this expression, one can theoretically estimate the reduction of the input that happens at the resonance of the equipment and use as a guideline for arriving at the notched input during design / testing of the equipment. While using this methodology, one need not measure the base force during the testing, which will be of great convenience.

2. BEHAVIOUR OF REDUCTION OF INPUT

Behaviour of the hard mounted equipment at its resonance frequency was like the behaviour in a single-degree-of-freedom (SDOF) system. To understand its behavior when mounted on an equipment panel, a system modeled as a two degree-of-freedom (2DOF), as shown in Fig. 1 was analysed. The equipment was represented by a SDOF system having parameters k_2, $m_2$and c_2 was considered to be mounted on a structure represented by a SDOF system having parameters k_1, m_1 and c_1. In these models, k, m and c represented stiffness, mass and damping coefficient. The mounting platform was generally a honeycomb sandwich panel. The usage of SDOF systems to represent the behavior was justified as the interest was around a certain frequency which was the fundamental mode of the equipment. The fundamental mode was represented by a SDOF system and the other modes were of no concern at the fundamental mode frequency.

DOF2 gave the response of the equipment and DOF1 gave the response at the interface of the equipment. One can consider that the loads for the equipment were derived considering it as a 2DOF system, while it was a SDOF system when the equipment was tested.

The differential equations of motion of the system were:

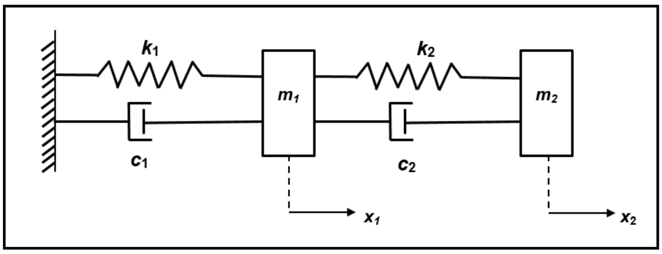

Figure 1. 2DOF system.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x_1} \\ \ddot{x_2} \end{pmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}; \quad (1)$$

and ω_1 and ω_2 were the natural frequencies of the 2DOF system. Here, ω_{11} and ω_{22} denoted the natural frequencies of the two SDOF systems that form the 2DOF system. Therefore, ω_{22} was the natural frequency of the equipment and ω_{11} was the natural frequency of the panel without the equipment. The value of ω_{11} was very high as it was the natural frequency of the panel without the equipment. The forces acting at the two degrees-of freedom were denoted by $f_1\,(t)\,, f_2(t)$ respectively.

2.1. A Typical Example

Consider an oscillator having a mass of 3 kg and fundamental natural frequency of around 400 Hz. The natural frequency can be of any value but 400 Hz was selected as the natural frequencies of many equipment lie between 200 to 600 Hz. Thus, $m_2 = 3$ kg, $k_2 = 2 \times 10^7$ N/m, $f_{22} = 411$ Hz, $\omega_{22} = 2582$ rad/s (corresponds to 411 Hz). Assume this oscillator was attached to a spring with a stiffness of 10^8 N/m. Hence $k_1 = 10^8$ N/m and m_1 was negligible. The natural frequencies of the combined system were $f_1 = 375$ Hz and $f_2 = 17435$ Hz. A random force having a PSD of $1.25 \times 10^{14}\,/f^4$ N^2/Hz was applied at DOF1. This amounted to a force of 4883 N^2/Hz at 400 Hz and produced an acceleration of 0.2 g^2/Hz at DOF1 at low frequencies. The force can be of any value, but the above value was selected so as to generate random vibrations close to some typical random responses at the interface which range from 0.1 to 0.4 g^2/Hz. Force was not applied on DOF2. This was justified by the fact[15] that when the equipment panel was subjected to acoustic excitation, the random vibrations experienced by the equipment were mainly from those transferred from the response of the panel than due to direct acoustic loading on the equipment.

Acceleration responses at DOF1 and DOF2 were computed. The response of DOF1 is shown in Fig. 2. There was a peak of 3.5 g^2/Hz in the PSD of response of DOF1, at around 375 Hz, which was at the natural frequency of the combined system. The oscillator had a response of 100 g^2/Hz at the same frequency, which is shown in Fig. 3.

The practice was to specify a PSD of 3.5 g^2/Hz for some band of frequency as the random vibration load of the equipment. If the equipment was subjected to the above excitation, it was expected to experience a response of 2200 g^2/Hz at its natural frequency i.e., 411 Hz whereas the response of the equipment when mounted on the panel was only 100 g^2/Hz (at 375 Hz). To circumvent this problem, the spacecraft industry

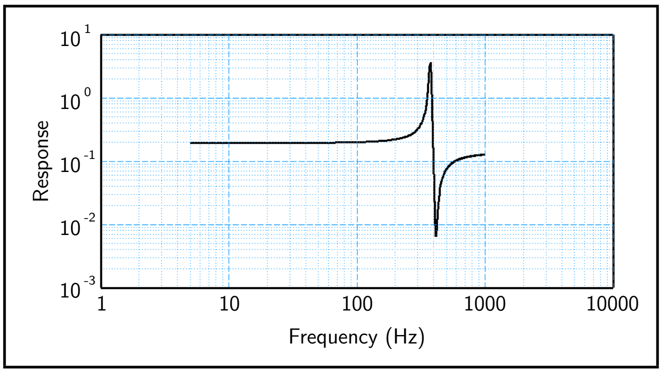

Figure 2. PSD of acceleration (g^2/Hz) at oscillator interface.

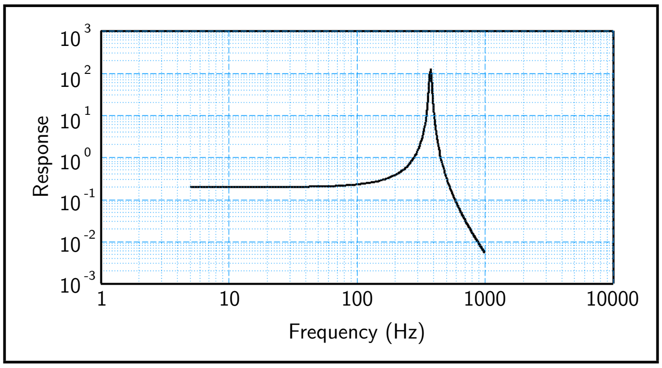

Figure 3. PSD of acceleration (g^2/Hz) of the oscillator.

resorts to the reduction of the input at the natural frequency of the equipment. This is referred to as notching such that the response of the equipment is not higher than the response of the equipment when mounted on the panel. In the example discussed, the input at 411 Hz was reduced from 3.5 g^2/Hz to 0.16 g^2/Hz, so that the response did not exceed 100 g^2/Hz.

It is possible to implement the notching of the input, provided the information on the response of the equipment (i.e. 100 g^2/Hz) when it is mounted on the panel is available. The rigorous method used was to estimate this response by using a finite element model. This necessitated developing the finite element model of the system and having several computations which involved a very significant amount of work and was therefore not practical. Since the same equipment needed to be used for various spacecraft, performing the testing for different levels catering to different spacecraft was also not suitable. In other words, arriving at the notched input without being sensitive to the details of the spacecraft was preferred. Though the force limited vibration testing method was widely used, it had some limitations as discussed before, such as the arriving at PSD of the required base force and the need for measuring the base force. Therefore, an alternate method for arriving at the notched input was investigated.

2.2. Reduction of Input

A close examination of the response of the panel (DOF1) showed that, though the response of DOF1 was very high near the natural frequency of the system (3.5 g^2/Hz at 375 Hz), the response of DOF1 at the natural frequency of the oscillator (411 Hz) was very low, 0.006 g^2/Hz. In other words, the response at the interface of the equipment was reduced at the resonance frequency of the equipment.

It was seen that if the natural frequency of the equipment

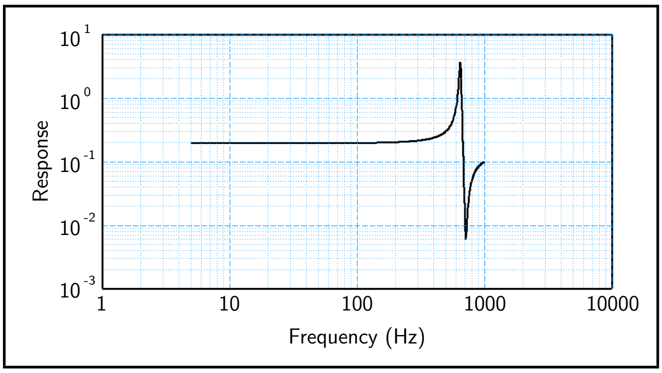

Figure 4. PSD of acceleration (g^2/Hz) at oscillator interface, m_2= 1 kg.

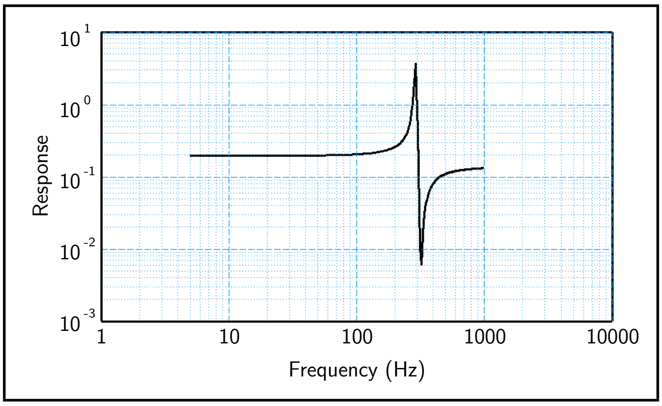

Figure 5. PSD of acceleration (g^2/Hz) at oscillator interface, m_2= 5 kg.

was changed, the frequency at which the response was reduced was also modified and it was always at the natural frequency of the equipment. This is demonstrated in Figs. 4 and 5. The response at DOF1 when the mass of the oscillator is 1 kg (natural frequency of the equipment being 700 Hz) is shown in Fig. 4. One can see that the reduction of the input to the oscillator happened at 700 Hz. When the mass of the oscillator was 5 kg (natural frequency of the equipment being 300 Hz), the reduction happened at the natural frequency of the equipment. Soucy and Cote[16] pointed to this behavior by drawing an analogy with a dynamic absorber, but no simulation or analytical treatment was presented. They considered base force for arriving at the notching and not the acceleration at the interface.

It is very important to note that in all the three cases the response at the interface of the equipment was reduced at the resonance frequency of the equipment. However, in all the three cases, the peak response at the interface of the equipment remained at 3.5 g^2/Hz, the response of the oscillator was 100 g^2/Hz and the input at the resonance frequency of the equipment was 0.006 g^2/Hz.

It was further seen that the notched input was very much influenced by the response of the oscillator. This was concluded by studying the responses estimated for various values of responses of the oscillator which is accounted in the calculations through changes in the damping factors. Summary of the results is given in Table 1. The mass of the oscillator was 3 kg. The higher the response, the higher the depth of notching. Here the depth of notching means the ratio of the nominal input to the notched input. Higher depth of notching means higher reduction in the input. Another important result was that the response of the oscillator was about 30 times that of the response at its interface.

Table 1. Influence of the oscillator response on the notched input, mounted on a stiff platform.

Damping factor	Amplification of oscillator	Response of oscillator (g^2/Hz)	Response at I/F of oscillator (g^2/Hz)	Notched input (g^2/Hz)	Depth of notching
0.01	50	450	15	0.002	7500
0.02	25	100	3.5	0.006	583
0.03	16.7	55	1.8	0.013	138

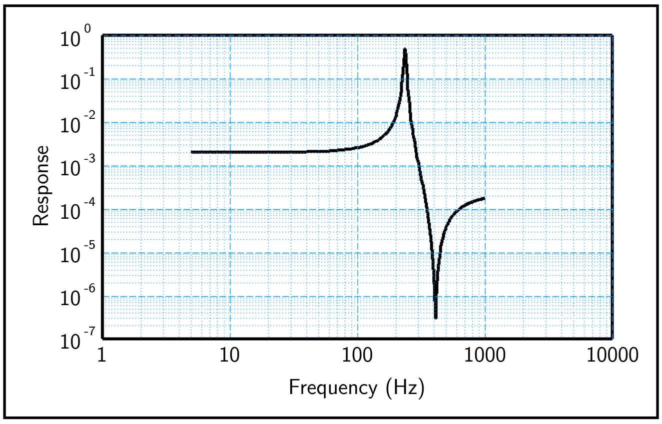

Figure 6. PSD of acceleration (g^2/Hz) at oscillator interface: flexible platform.

Thus, it is seen that at the resonance frequency of the equipment the response at the base of the equipment reduces if the equipment were mounted on a panel. The depth of reduction was higher for equipment having higher response, but it was insensitive to the mass of the equipment. This behavior can be effectively used in arriving at the notched input.

2.3. Influence of Platform Stiffness

All the above responses were seen when the equipment was mounted on a stiff platform. The behavior if the platform was more flexible than the equipment is now examined. The stiffness of the oscillator was $k_2 = 2 \times 10^7$ N/m and it was mounted on a spring with a stiffness of $k_1 = 1 \times 10^7$ N/m. For the above system $m_2 = 3$ kg, $f_{22} = 411$ Hz, $\omega_{22} = 2582$ rad/s (corresponds to 411 Hz). The natural frequencies of the system were $f_1 = 237$ Hz and $f_2 = 8724$ Hz.

A random force was applied at DOF1 and the response of the oscillator and the response at the base of the oscillator were determined. The force applied in this case was such that it produced an acceleration of 0.2 g^2/Hz at DOF1 without the presence of the oscillator (equipment). To achieve the above response, a force spectral density of $1.25 \times 10^{10} / f^4$(in N^2/Hz) was applied. This amounted to a force of 0.5 N^2/Hz at 400 Hz.

The behaviour was identical to that seen when the equipment was mounted on a stiff platform. The responses were determined with the oscillator having a mass of 1 kg and then 5 kg. In all the three cases, the peak response at the interface of the equipment was 0.5 g^2/Hz, the response of the oscillator was 1 g^2/Hz and the notched input at the resonance frequency of the equipment was 3×10^{-7} g^2/Hz. Figures showing these responses are not given here for brevity. Results show that the depth of notching was higher if the response of the equipment was higher. It can also be seen that the response of the oscillator was about 2 times that of the response at its interface.

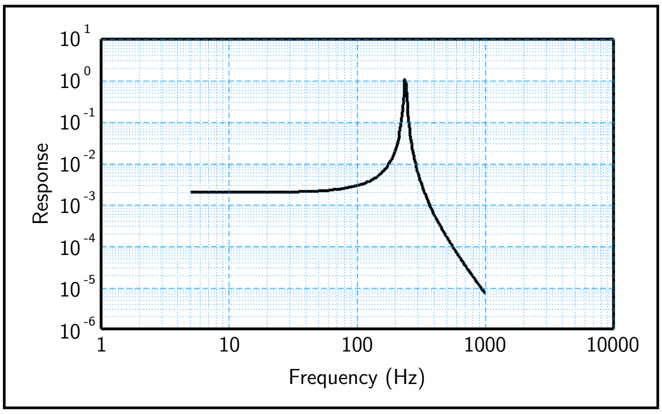

Figure 7. PSD of acceleration (g^2/Hz) of oscillator: flexible platform.

3. AN EXPRESSION FOR NOTCHED INPUT

The results presented before are based on the specific values of the systems and raises the question of whether this phenomenon happens in all situations. Therefore, an analytical expression for the notched input is derived here, without assigning any specific value for the parameter of the system. Also, it will be very convenient if an expression is available, as one need not do simulation for each parameter of the system. Numerical simulation alone does not give an insight in to the behavior whereas the analytical expression will provide.

Consider the system shown in Fig. 1. The differential equations of motions are as given by Eq. (1).

One can determine the response of the system either by direct method with damping represented through complex terms or using the modal superposition principle. The first method expresses the response in terms of physical parameters of the system and hence they can be expressed in terms of parameters of the independent SDOF system like natural frequency of the equipment etc. The second method expresses the response in terms of modal parameters of the system and they can not be expressed in terms of parameters of the independent SDOF system. Therefore, the first method is used in this work.

3.1. Response of DOF1 at $\omega=\omega_{22}$

Consider a force excitation of $F_1 e^{j\omega t}$. The amplitude of the displacement response of DOF1 at frequency ω can be obtained by solving Eq. (1) as:

$$X_1 = \frac{\left\{1-(\frac{\omega}{\omega_{22}})^2 + j\frac{c_2}{k_2}\omega\right\}}{A(\omega)}\frac{F_1}{k_1}; \tag{2}$$

where A is defined by:

$$A(\omega) = \left\{1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_{11}}\right)^2 + j\frac{c_1+c_2}{k_1}\omega\right\} \cdot \left\{1-\left(\frac{\omega}{\omega_{22}}\right)^2 + j\frac{c_2}{k_2}\omega\right\} - \frac{k_2}{k_1}\left\{1+j\frac{c_2}{k_2}\omega\right\}^2. \tag{3}$$

Correspondingly the amplitude of displacement response of DOF2 is:

$$X_2 = \frac{\left\{1+j\frac{c_2}{k_2}\omega\right\}}{A(\omega)}\frac{F_1}{k_1}. \tag{4}$$

At $\omega=\omega_1$, that is at the natural frequency of the system, there is a significant response of X_1. As discussed in chapter 2, at $\omega=\omega_{22}$, which is the natural frequency of the equipment, there is a reduction in the response of X_1 (notch). The objectives is to determine the response of X_1 at the natural frequency of the equipment. At $\omega=\omega_{22}$, from Eq. (1),

Table 2. Influence of the oscillator response on the notched input, mounted on a flexible platform.

Damping factor	Amplification of oscillator	Response of oscillator (g^2/Hz)	Response at I/F of oscillator (g^2/Hz)	Notched input (g^2/Hz)	Depth of notching
0.01	50	2.7	1	1 E-7	1.0 E7
0.02	25	1	0.5	3 E-7	0.2 E7
0.03	16.7	0.5	0.2	7 E-7	0.03 E7

$$X_1 = \frac{j(c_2/k_2)\omega}{A\left(\omega_{22}\right)}\frac{F_1}{k_1}; \tag{5}$$

with A at $\omega=\omega_{22}$ being:

$$A(\omega_{22}) = \left\{1+\frac{k_2}{k_1}-\left(\frac{\omega_{22}}{\omega_{11}}\right)^2 + j\frac{c_1+c_2}{k_1}\omega\right\} j\frac{c_2}{k_2}\omega - \frac{k_2}{k_1}\left\{1+j\frac{c_2}{k_2}\omega\right\}^2. \tag{6}$$

Assuming $\omega_{11} \gg \omega_{22}$, which is true in most of the situations, $A(\omega_{22})$ is:

$$A(\omega_{22}) = \left\{1+\frac{k_2}{k_1} + j\frac{c_1+c_2}{k_1}\omega\right\} j\frac{c_2}{k_2}\omega - \frac{k_2}{k_1}\left\{1+j\frac{c_2}{k_2}\omega\right\}^2. \tag{7}$$

Collecting all the real parts and imaginary parts:

$$A\left(\omega_{22}\right) = \left\{\frac{{c_2}^2\omega^2}{k_1 k_2} - \frac{(c_1+c_2)c_2}{k_1 k_2}\omega^2 - \frac{k_2}{k_1}\right\} + j\left\{\frac{c_2\omega}{k_2} - \frac{c_2\omega}{k_1}\right\}. \tag{8}$$

On simplification:

$$A\left(\omega_{22}\right) = \left\{-\frac{c_1 c_2}{k_1 k_2}\omega^2 - \frac{k_2}{k_1}\right\} + jc_2\omega\left\{\frac{1}{k_2}-\frac{1}{k_1}\right\}. \tag{9}$$

Therefore, the displacement X_1at $\omega=\omega_{22}$ is:

$$X_1 = \frac{j(c_2/k_1 k_2)\omega F_1}{\left\{-\frac{c_1 c_2}{k_1 k_2}\omega^2 - \frac{k_2}{k_1}\right\} + jc_2\omega\left\{\frac{1}{k_2}-\frac{1}{k_1}\right\}}; \tag{10}$$

which is simplified as:

$$X_1 = \frac{jc_2\omega F_1}{-\left\{{k_2}^2 + c_1 c_2\omega^2\right\} + jc_2\omega\left\{k_1-k_2\right\}}. \tag{11}$$

In a convenient form, the amplitude of responseX_1 at $\omega=\omega_{22}$ can be written as (in complex form):

$$X_1 = \frac{-jF_1}{\left\{\frac{{k_2}^2}{c_2\omega} + c_1\omega\right\} + j\left\{k_2-k_1\right\}}. \tag{12}$$

Equation (12) gives the response of the mounting platform at the interface of the equipment at the natural frequency of the equipment.

3.2. Response of DOF1 at $\omega = \omega_1$

The differential equations of motion of a 2DOF system, in terms of generalized coordinates denoted by q_1, q_2 are:

$$\ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + {\omega_1}^2 q_1 = \overline{f_1}; \tag{13}$$

$$\ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + {\omega_2}^2 q_2 = \overline{f_2}; \tag{14}$$

where $\overline{f_1}$ and $\overline{f_2}$ are the generalized forces, which in this case are:

$$\begin{Bmatrix} \overline{f_1} \\ \overline{f_2} \end{Bmatrix} = \begin{bmatrix} {u_1}^1 & {u_2}^1 \\ {u_1}^2 & {u_2}^2 \end{bmatrix} \begin{Bmatrix} f_1 \\ 0 \end{Bmatrix}; \tag{15}$$

where $\begin{Bmatrix} {u_1}^1 \\ {u_1}^2 \end{Bmatrix}$ and $\begin{Bmatrix} {u_2}^1 \\ {u_2}^2 \end{Bmatrix}$ are the modal vectors of the system. The aim is to determine the response of DOF1 at $\omega = \omega_1$.

As per modal superposition theorem:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} {u_1}^1 & {u_2}^1 \\ {u_1}^2 & {u_2}^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}. \tag{16}$$

The frequency response function of the natural coordinate q_1 at $\omega = \omega_1$ is $(1/2\zeta_1)$. Thus:

$$q_1 = \frac{{u_1}^1 f_1}{2\zeta_1{\omega_1}^2}. \tag{17}$$

In practical cases, since ω_2 is well separated from ω_1, the response of the natural coordinate q_2 at $\omega = \omega_1$ can be neglected.

Therefore, the amplitude of response of DOF1 at $\omega = \omega_1$ is:

$$X_1 = \frac{\{{u_1}^1\}^2 F_1}{2\zeta_1{\omega_1}^2}. \tag{18}$$

Equation (18) gives the response of the mounting platform at the interface of the equipment at the natural frequency of the system.

3.3. Mean Square Value of Response of DOF1

We can now express the response at the interface of the equipment (DOF1) at the natural frequency of the equipment in terms of the peak response that occurred at $\omega = \omega_1$ by combining Eqs. (12) and (18) as:

$$\frac{X_{1_{\omega=\omega_{22}}}}{X_{1_{\omega=\omega_1}}} = \frac{-j}{\left\{({k_2}^2/c_2\omega) + c_1\omega\right\} + j\ \{k_2 - k_1\}} \frac{2\zeta_1{\omega_1}^2}{\{{u_1}^1\}^2}. \tag{19}$$

One can now determine the mean square values of the responses for a random excitation. As there was no excitation at DOF2, the cross-spectral density of the excitation vanished and the mean square value of the response is obtained as[17]:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{\left\{({k_2}^2/c_2\omega) + c_1\omega\right\}^2 + \{k_2 - k_1\}^2} \frac{4{\zeta_1}^2{\omega_1}^4}{\{{u_1}^1\}^4}. \tag{20}$$

It will be convenient to express the above relation in terms of a few important parameters, such as k_1/k_2, m_2 and ζ_{22}. Thus,

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{\left\{\frac{1}{2\zeta_{22}} + 2\frac{c_1}{c_2}\zeta_{22}\right\}^2 + \left\{1 - \frac{k_1}{k_2}\right\}^2} \frac{4{\zeta_1}^2{\omega_1}^4}{{k_2}^2\{{u_1}^1\}^4}. \tag{21}$$

Denoting $\frac{1}{2\zeta_{22}}$ by Q_{22} and since the second term is negligible in comparison with the first term, we get:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{\{Q_{22}\}^2 + \left\{1 - \frac{k_1}{k_2}\right\}^2} \frac{4{\zeta_1}^2{\omega_1}^4}{{k_2}^2\{{u_1}^1\}^4}. \tag{22}$$

Equation (18) gives the response of the mounting platform at the natural frequency of the equipment (which is the notched input) in relation to the response at the natural frequency of the system.

3.4. Mass Normalized Modal Vector

In Eq. (22), ${u_1}^1$ was mass normalized modal vector. It will be convenient if the modal vector in Eq. (22) is expressed in terms of other parameters. Assume that m_1 was negligible. Applying the equation of motion to DOF1 leads to:

$$x_1 = \frac{k_2}{k_1 + k_2} x_2. \tag{23}$$

Thus, the modal vector is $\begin{Bmatrix} \frac{k_2}{k_1+k_2} x_2 \\ x_2 \end{Bmatrix}$. The modal vector x_2 should be such that it is mass normalized. Therefore, the modal mass should be unity. Hence:

$$\left\{ m_2 + \frac{m_1}{\left\{1 + \frac{k_1}{k_2}\right\}^2} \right\} {x_2}^2 = 1; \tag{24}$$

$$x_2 = \left\{ m_2 + \frac{m_1}{\left\{1 + \frac{k_1}{k_2}\right\}^2} \right\}^{-1/2}. \tag{25}$$

Denoting $\left\{ m_2 + \frac{m_1}{\left\{1 + \frac{k_1}{k_2}\right\}^2} \right\}^{-1/2}$ by β, the mass normalized modal vector is $\begin{Bmatrix} \frac{1}{1+(k_1/k_2)}\beta \\ \beta \end{Bmatrix}$.

3.5. Response of Mounting Platform

Substituting the modal vector given above into Eq. (22), we get:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{\{Q_{22}\}^2 + \left\{1 - \frac{k_1}{k_2}\right\}^2} \frac{4{\zeta_1}^2{\omega_1}^4}{{k_2}^2\left\{1 + \frac{k_1}{k_2}\right\}^{-4}\beta^4}; \tag{26}$$

which can be modified as:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{\{Q_{22}\}^2 + \left\{1 - \frac{k_1}{k_2}\right\}^2} \frac{1}{{Q_1}^2} \frac{1}{{m_2}^2} \left\{\frac{\omega_1}{\omega_{22}}\right\}^4 \left\{1 + \frac{k_1}{k_2}\right\}^4 \beta^{-4}. \tag{27}$$

In most cases, since ω_1 is close to ω_{22}, Eq. (27) reduces to:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{\{Q_{22}\}^2 + \left\{1 - \frac{k_1}{k_2}\right\}^2} \frac{1}{{Q_1}^2} \frac{1}{{m_2}^2} \left\{1 + \frac{k_1}{k_2}\right\}^4 \beta^{-4}. \tag{28}$$

Though the equation is written in terms of displacements, the same relation holds good for acceleration. Equation (28) can be used in determining the response of the mounting platform at the natural frequency of the equipment which can form the input that needs to be applied during the random vibration testing of the equipment. The response is expressed as a function of system parameters and does not correspond to any specific value of the system parameter.

4. CHARACTERISTICS OF THE NOTCHED INPUT

As mentioned previously, Eq. (28) can be used in determining the notched input. The response ${X_1}^2_{\omega=\omega_1}$ gets reduced to ${X_1}^2_{\omega=\omega_{22}}$ by a factor given by Eq. (28). The factor is less than unity and dependent on the ratio of stiffness values, mass and amplification of the equipment. How the notched input varies with these parameters of the system is analysed.

For easy understanding, the expression is simplified for specific cases of practical interest and the influence of the system parameters is discussed.

4.1. Platform Stiffness Very High

4.1.1. Expression for the notched input

The expression can be simplified when the panel stiffness is very high compared to that of the equipment. When $\frac{k_2}{k_1} \ll 1$, $\beta = \left\{m_2 + \frac{{k_2}^2}{{k_1}^2} m_1\right\}^{-1/2}$:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{{Q_{22}}^2 + \frac{{k_1}^2}{{k_2}^2}} \frac{1}{{Q_1}^2} \frac{1}{{m_2}^2} \frac{{k_1}^4}{{k_2}^4} \left\{m_2 + \frac{{k_2}^2}{{k_1}^2} m_1\right\}^2. \tag{29}$$

This can be further modified as:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{{Q_{22}}^2 + \frac{{k_1}^2}{{k_2}^2}} \frac{1}{{Q_1}^2} \frac{{k_1}^4}{{k_2}^4} \left\{1 + \frac{{k_2}^2}{{k_1}^2} \frac{m_1}{m_2}\right\}^2. \tag{30}$$

Equation (30) allows estimation of the notched input.

A simplified form of Eq. (30) can be arrived at since $\frac{k_2}{k_1} \ll 1$ as:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{{Q_{22}}^2 + \frac{{k_1}^2}{{k_2}^2}} \frac{1}{{Q_1}^2} \frac{{k_1}^4}{{k_2}^4}. \tag{31}$$

The depth of notching denoted by δ, which is the ratio of the un-notched input to the notched input, (reciprocal of the above expression) is given by:

$$\delta = ({Q_{22}}^2 + \frac{{k_1}^2}{{k_2}^2})\, {Q_1}^2 (k_2/k_1)^4. \tag{32}$$

Equation (32) can be used in arriving at the input at the natural frequency of the equipment if the panel stiffness is quite high compared to the equipment. It is to be noted that the parameter δ is the ratio of spectral densities which is the square of the reduction in acceleration and hence will be a high value. The reduction in acceleration is the square root of it.

A further simplification of Eq. (32) can be made, when Q_{22} is quite high compared to $\frac{k_2}{k_1}$, as:

$$\delta = {Q_{22}}^2\, {Q_1}^2 (k_2/k_1)^4. \tag{33}$$

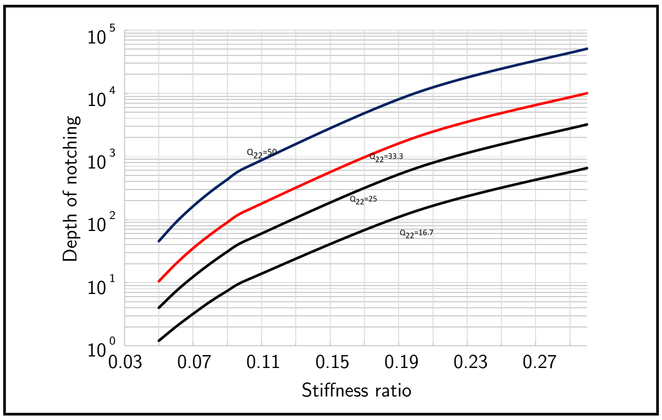

Figure 8. Depth of notching for various values of the system parameters.

4.1.2. Characteristics of the notched input

Depth of notching depends mainly on 3 parameters. Those parameters are the amplification in the equipment (Q_{22}), amplification in the first mode of the system (Q_1) and the ratio of the stiffness values k_2/k_1. The dependence of the depth of notching on Q_1 is straight forward and it is proportional to ${Q_1}^2$. Dependence of the depth of notching on the other two parameters is shown in Fig. 8. The results are shown for values of k_2/k_1 varying from 0.03 to 0.3 for various values of Q_{22} such as 16.7, 25, 33.3 and 50 which span the practically observed amplifications. The value of Q_1 is taken to be 16.7. Equation (32) is valid for values of $k_2/k_1 << 1$ and therefore the curves are drawn for values of k_2/k_1 only up to 0.3. It does not mean that the present method cannot be used for determining the notched input for other values of k_2/k_1. In such cases, eq. (28) needs to be considered.

The depth of notching of the input can be obtained from the curves shown in Fig. 8, for the values of certain parameters of the system. One can see that higher the amplification of the oscillator higher the depth of notching. Higher the stiffness ratio (k_2/k_1) the depth of reduction in the input also will be higher. Flexibility of the platform only brings down the input. If the platform stiffness is significantly high, the system behaves as in a hard mounted situation and no reduction in input shall occur. This behavior is also exhibited by the notched input obtained through the method developed here. For very low values of k_2/k_1, that is the platform stiffness is quite high, the expression derived in this work does not suggest a reduction in the input.

4.1.3. Comparison with the results of numerical simulation

Let us now compare the results obtained using the expression derived here with the results obtained using the exact solution for a 2DOF system. This is done by creating a 2DOF system and solving for random response using NASTRAN which can be taken as the reference. Consider the case when the oscillator (equipment) is mounted on a stiff platform, as considered in chapter 3. The stiffness of the oscillator is $k_2 = 2\times10^7$ N/m and it is mounted on a spring having stiffness $k_1 = 10 \times 10^7$ N/m ($k_2/k_1 = 0.2$). The mass of the oscillator is 3 kg. The depth of the notching is compared with that estimated through the expression derived in this work, given in Table 3. For the damping factor in the range of practical values (0.02 to 0.03), the expression derived in this work can give an

Table 3. Results for oscillator on stiff platform.

Damping factor	Q_{22}	Response at I/F of oscillator (g^2/Hz)		Depth of notching		
		at ω_1	at ω_{22}	numerical simulation	present method	force limited testing
0.01	50	15	0.002	7500	10100	2500
0.02	25	3.5	0.006	583	650	625
0.03	16.7	1.8	0.013	138	136	279

accurate estimate of the depth of notching. The amplification of 50 is an extreme case and rarely occurs.

4.1.4. Comparison with the results of force limited testing

It is interesting to compare the notched input determined using the present methodology with the input if the existing force limited testing was used. As mentioned before, product of the square of the mass and the PSD of the unnotched input is considered as the PSD of the force to be limited. Thus, the force to be limited is $m_2\times$unnotched input. At the resonance frequency of the equipment the force generated at the base will be $m_2 {Q_{22}}^2\times$notched input. Based on the above arguments it can be shown that during force limited testing

$$\delta = {Q_{22}}^2. \tag{34}$$

Comparison of Eq. (32) with Eq. (34) will reveal the difference in the notched input determined using the proposed method and that applied in a force limited testing. Results for a few cases are given in Table 3. The input worked through both the methods could be the same under certain conditions, otherwise they are different as can be seen from Eqs. (32) and (34). As mentioned before, the force limited testing considers only the mass of the equipment in arriving at the force spectrum to be limited. The present method considers influence of many other parameters of the system in determining the notched input. Also, the present method does not seek the measurement of force.

4.2. Platform Stiffness Very Low

The expression can be simplified when the panel stiffness is very low compared to the equipment. When $\frac{k_1}{k_2} \ll 1$, $\beta = \{m_2 + m_1\}^{-1/2}$:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{{Q_{22}}^2}\frac{1}{{Q_1}^2}\frac{1}{{m_2}^2}\{m_2+m_1\}^2; \tag{35}$$

which can be further modified as:

$$\frac{{X_1}^2_{\omega=\omega_{22}}}{{X_1}^2_{\omega=\omega_1}} = \frac{1}{{Q_{22}}^2} * \frac{1}{{Q_1}^2}\left\{1+\frac{m_1}{m_2}\right\}^2. \tag{36}$$

The depth of notching will be:

$$\delta = {Q_{22}}^2{Q_1}^2\{1+(m_1/m_2)\}^{-2}. \tag{37}$$

Equation (37) can be used in arriving at the input at the natural frequency of the equipment if the panel stiffness is quite low compared to the equipment.

In this case also the depth of notching depends mainly on 3 parameters, that is the amplification in the equipment (Q_{22}), amplification in the first mode of the system (Q_1) and the ratio of the masses m_1/m_2. As the dependence of the depth of notching on these parameters is a straightforward relation, the curves showing their variations are not presented. The higher the amplification of the oscillator, higher the depth of notching. Results like those obtained for platform with high stiffness are obtained as both are derived from the same original expression, given by Eq. (28).

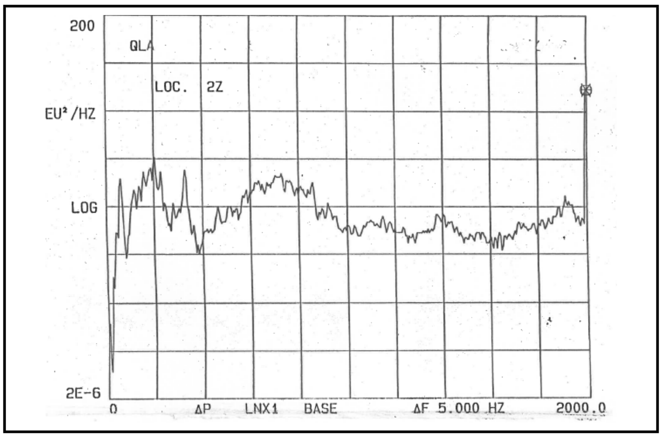

Figure 9. PSD of acceleration response at the interface of the equipment.

5. AN APPLICATION

Usage of the above results in determining the notched input for equipment is demonstrated with an example. The response measured at the interface of equipment is shown in Fig. 9. A random vibration load of 0.2 g^2/Hz in the frequency range 100 Hz to 500 Hz would have been specified, based on the measured acceleration.

The mass of the equipment is 4.5 kg and its natural frequency is 390 Hz. With a damping factor of 0.03, the expected amplification at certain location is 16.7. Application of an acceleration of 0.2 g^2/Hz will result in a response of 56 g^2/Hz at the above location. The equipment designer / manufacturer finds that it can withstand a response of only 1.5 g^2/Hz which translates to an input of 0.0036 g^2/Hz.

Using Eq. (32), the expected reduction in the input at the resonance frequency of the equipment can be calculated. As the equipment has several modes, the value of m_2 should not be the entire mass, instead it should be the effective mass in the fundamental mode which is available as part of free vibration analysis. For this equipment this is determined as 0.9 kg. The value of k_2 is 0.54×10^7 N/m which is the stiffness of the equipment. The stiffness of a typical mounting platform is obtained theoretically as 2.5×10^7 N/m and the stiffness ratio is 0.216. Using these parameters, the depth of reduction in the input at the natural frequency can be calculated as 182. One can determine the expected input at 390 Hz (natural frequency of the equipment) as 0.001 g^2/Hz and therefore allowing the input to be 0.005 g^2/Hz is acceptable. One can consider that the results obtained based on this work gives the minimum input that needs to be applied and any value above this is acceptable. It is also interesting to note that the measured acceleration at the interface of the equipment in this case (shown in Fig. 9) is about 0.002 g^2/Hz at around 390 Hz. The response significantly drops from 0.16 g^2/Hz to about 0.002 g^2/Hz at around 390 Hz. If the existing methodology of force limited testing is employed, the PSD of the base force to be limited would be

390 N^2/Hz ($4.5^2 \times 0.2 \times 9.81^2$) and to limit the above value of force, an input of 0.02 g^2/Hz will be applied at the resonance frequency.

While using the present method, the parameters required to arrive at the notched input are the natural frequency of the equipment, effective mass in that mode of vibration and the damping factor. All these parameters are the characteristics of the equipment. Required parameters about the mounting platform are the damping factor (could be taken as 0.03) and the stiffness of the mounting platform at the equipment interface. In this methodology the base force that can be limited need not be known and the base force need not be measured.

6. SUMMARY AND CONCLUSIONS

It is shown that the response at the interface with the mounting platform gets reduced at the resonance frequency of the equipment. An expression to estimate this reduction of the acceleration is analytically derived which is expressed in terms of various parameters of the system. Results using this expression are compared with the exact solution using NASTRAN. The results show that the derived expression can very well be used in determining the notched input, except at very low values of damping.

Reduction of the input at the resonance frequency of the equipment increases with the increase in the response of the oscillator. The reduction of the input is sensitive to the stiffness ratios rather than the mass ratios of the oscillator and the mounting panel. Higher the stiffness ratio (k_2/k_1) the depth of reduction in the input also will be higher. The flexibility of the platform only causes the reduction of the input. If the platform stiffness is significantly high, there will not be any reduction in the input. The notched input that results in during the force limit method is generally different from those obtained from exact solution as only the mass of the equipment is considered for arriving at the force limits. Though the present methodology needs information of several structural parameters, the notched input estimated is close to the actual.

The results of the present work help in arriving at suitable notching of the input acceleration during the random vibration tests on equipment, without needing any complex force measurement system.

ACKNOWLEDGEMENT

Some of the ideas were evolved during the author's discussions with Dr. P. S. Nair, earlier Group Director of Structures Group, UR Rao satellite Centre. Author wishes to acknowledge the help offered by Ms. Josephine Kelvina Florence of Structures Group for performing some of the simulations.

REFERENCES

[1] Girard, A. and Moreau, D. Derivation of satellite equipment test specifications from vibration and acoustic test data, *ESA Journal*, **10**, 311–321 (1986).

[2] Girard,A. and Moreau, D. Elaboration of a continuous function of unit mass for vibration testing, *ESA Journal*, **11**, 83–89 (1987).

[3] Saltor, J.P. Advances in Numerology, *Shock and Vibration Digest*, **23**, 3–8 (1991).

[4] Scharton, T.D. *Force limited vibration testing monograph*, NASA RP-1403 (1997).

[5] Girard, A., Dupuis, P.E. and Bugeat, L.P. Notching in random vibration testing. *Proc. of European Conference on Spacecraft Structures, Materials and Mechanical Testing*, Germany, November 4–6, ESA SP-428 (1998). ISBN: 92-9092-7127.

[6] Girard, A. and Newerla, A. Methodology for notching in random vibrations testing. *Proc. of 4 th International Symposium on Environmental Testing for Space Programmes, Materials and Mechanical Testing*, Belgium, June 12–14, ESA SP-467 (2001) ISBN:92-9092-7097.

[7] Himelblau, H., Manning, J.E., Piersol, A.G. and Rubin, S. *Dynamic environmental criteria*. NASA Technical Handbook, NASA-HDBK-7005 (2001).

[8] Scharton, T.D. and Kolaini, A.R. *Updates to Force Limited Vibration Testing*. NASA Technical Handbook, NASA-HDBK-7004C (2017).

[9] Chang, K.Y. and Scharton, T.D. Cassini spacecraft and instrument force limited vibration testing. *Proc. of the 3rd International Symposium on Environmental Testing for Space Programms*, Netherlands, June 24–27, ESA SP-408 (1997).

[10] Chang, K.Y. and Scharton, T.D. Cassini spacecraft force limited vibration testing, *Sound and Vibration*, **48**, 16-20 (1998).

[11] Salvignol, J.C., Laine, B., Ngan, I., Honnen, K. and Kommer, A. Notching during random vibration test based on interface forces - The JWST NIRSPEC experience. *Proc. of 11 th European Conference on Spacecraft Structures, Materials and Mechanical Testing*, France, September 15–17, ESA SP-428 (2009).

[12] Soucy, Y. and Yvan. On Force Limited Vibration for Testing Space Hardware. *Proc. of the IMAC XXIX Conference*, Jan. 31 – Feb. 3 (2011). Appeared in *Advanced Aerospace Applications*, **1**, 63–71, 2011 https://dx.doi.org/10.1007/978-1-4419-9302-1_6.

[13] Calvi, A. Reduction of overtesting during base driven random vibration tests for the Euclid spacecraft hardware. *Proc. of 4th ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering* (COMPDYN 2013), Greece, June 12–14 (2013).

[14] Wijker, J.J, Ellenbroek, M.H.M and Boeri de A. Force limited random vibration testing: the computation of the semi-empirical constant C2 for a real test article and unknown supporting structure, *CEAS Space Journal*, **7**(3), 359–373 (2015). https://dx.doi.org/10.1007/s12567-015-0086-0.

[15] Renji, K. Application of Statistical Energy Analysis (SEA) in estimating acoustic response of panels with non-uniform mass distribution, *Int. J. of Acoustics and Vibration*, **26** (1), 80–87, (2021). https://dx.doi.org/10.20855/ijav.2020.25.11736.

[16] Soucy, Y. and Cote, A. Reduction of overtesting during vibration tests on space hardware. *Canadian Aeronautics and Space Journal*, **48**, 77–86, (2002). https://dx.doi.org/10.5589/q02-006.

[17] Nigam, N.C. and Narayanan, S. *Application of Random Vibrations,* Narosa Publishing House, New Delhi (1992).