Determination of the Fundamental Lateral Bending Mode Frequency of a Spacecraft from its Static Deflection

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The primary structure of a spacecraft that involves the central cylinder is designed to meet a specific value of fundamental lateral mode frequency. Though this frequency can be theoretically estimated by using a finite element model, its experimental verification is only possible when the entire spacecraft is made ready. However, static load tests are done on the primary structure of the spacecraft as soon as the structure is realized. In this work, a methodology to determine the fundamental bending mode frequency of a spacecraft from the deflections measured when the structure alone is subjected to static load tests is developed. An expression relating the fundamental lateral bending mode frequency of a spacecraft, having several masses attached at different locations, and the deflection at its tip is derived. Two spacecraft, one of 3000 kg class and the other of 6000 kg class are taken as examples. The fundamental lateral frequencies are determined using this methodology. These frequencies are also obtained through resonance search tests. It is shown that the derived expression will be very useful in estimating the fundamental lateral mode frequency of a spacecraft with the help of the measured deflection.

NOMENCLATURE

a	Length of the beam beyond the attached mass
b	Length of the beam on which uniformly distributed mass is acting
E	Young's modulus of the material
g	Acceleration due to gravity
I	Area moment of inertia of the cross section
l	Length of the beam
$l_1,\ l_2$	Distances to the masses M_1 and M_2
M	Attached concentrated mass
$M_1,\ M_2$	Attached concentrated masses at distances l_1 and l_2
α	A parameter in relating frequency with deflection
δ_{tip}	Deflection at the tip
ω_n	Natural frequency in rad/s
ω_1	Fundamental natural frequency in rad/s
$\omega_{11},\ \omega_{22}$	Natural frequencies when only M_1 is present, only M_2 is present
ρ	Mass per unit length of the beam

1. INTRODUCTION

It is well known that the natural frequency of a single-degree-of-freedom system is directly related to its deflection caused by a load that corresponds to its weight. This principle can be used for determining the fundamental bending mode frequency of a beam having negligible self-mass with a mass attached at some location. If it is a cantilever having a mass at the tip, the natural frequency of the fundamental bending mode, denoted by ω_n, can be shown to be[1]:

$$\omega_n{}^2 = \frac{g}{\delta_{\text{tip}}}; \tag{1}$$

where δ_{tip} is the deflection at the free end of the cantilever when a static force due to the acceleration of 1 g of the mass is applied.

In most of the practical cases, the masses are attached at several locations on the beam, consequently static loads act at various locations, and the above relation is not valid. The natural frequencies of all such cases can be represented in the following general form[2,3]:

$$\omega_n{}^2 = \alpha \frac{g}{\delta_{\text{tip}}}; \tag{2}$$

with parameter α being different for different loading conditions. The value of α is 1.0 for a cantilever with tip mass. It can be shown that $\alpha = 1.55$ for a cantilever with a uniformly distributed load (UDL).[2]

In spacecraft having a central cylinder, the fundamental lateral mode of vibration involves the first bending mode of the central cylinder and the theories concerning vibrations of beams can be applied for this mode.[3,4] In a simple form, this mode can be considered like the fundamental bending of a cantilever, the main cylinder of the spacecraft being the cantilever. The design of the central cylinder is to achieve the fundamental lateral bending mode frequency greater than a specified value, for example 10 Hz. The thickness of the cylinder, and hence the mass of the cylinder, is decided by the value of the fundamental lateral bending mode frequency. Therefore, an accurate estimation of the frequency of this mode is essential for an optimum design of the cylinder.

The frequency of the bending mode can be theoretically estimated using a finite element model. As the spacecraft is an assemblage of several elements, the accuracy of the prediction depends on how accurately the flexibilities of these joints are represented in the finite element model. The frequency of this mode can be experimentally obtained through a resonance search test on the spacecraft. However, the test can be performed only when the entire spacecraft is built. This is too late in the spacecraft life cycle. Any experimental methodology that can give a close estimate of the natural frequency, without the entire spacecraft being available but flexibilities of all these joints included would be well appreciated. The frequency of the spacecraft will be verified at a later stage through dynamic tests.

Well before the entire spacecraft is assembled, the spacecraft structure alone undergoes a static load test. Since the lateral bending mode frequency is related to the static deflection, it is prudent to expect that it should be possible to estimate the frequency of the spacecraft from the deflections measured while static loads that correspond to 1 g are applied on the structure. The most important parameter required to estimate the natural frequency from its static deflection is the value of the parameter α for the given load distribution. As presented earlier, the values of this parameter are reported for a concentrated mass at the tip ($\alpha = 1.0$) and for a uniform mass along the cantilever ($\alpha = 1.55$). The spacecraft will have several concentrated masses at different locations along with some mass uniformly distributed over the structure. There are no works reported on the values of α for cantilevers with such mass distributions. Such relations are not existing except for simple cases like cantilever with tip mass etc. In the absence of the values of α for such mass distributions, one uses the value of 1.0 which leads to large errors in the estimated values of frequencies. Hence, this method is not suitable for determining the natural frequency of the spacecraft, unless the value of α can be determined. The present work aims at bridging this gap.

It is well known that the fundamental lateral bending mode is directly related to the deflection due to static loads. There are several works that relate the natural frequency with the static deflection. Bert,[5] Chai and Low[6] and Low[7] presented such relations for beams and Jones,[8] Bert[5] and Radhakrishnan et. al.,[9] investigated them for plates. It is to be noted in the Rayleigh-Ritz method[10,11] for determining the natural frequency[10–13] the function assumed is close to the static deflection shape, indicating the possible relation between the natural frequency and the static deflection. Such relations are usefully applied in many applications such as cantilevered sensors[14] etc. Bert[5] proposed such relations for determining the natural frequencies of several systems encountered in practice. They included a system with several identical springs and masses connected in a series, axial vibration of a rod having a mass at the tip etc. The influence of boundary conditions was also investigated by Bert.[5] Their study showed that α varies between 1.27 and 1.55, the latter being for the cantilever. It is to be noted that the beam in these studies has a uniform mass and there are no concentrated masses. Works on relations for beams with several concentrated masses are not reported.

The objective of the present work is to establish the relation between the frequency of the fundamental bending mode of a spacecraft and the deflection at its tip under static loads. The spacecraft is a beam with several masses attached at various locations along a UDL. An expression to determine the parameter α is derived. Once the parameter α is determined, the frequency of the fundamental mode can be estimated using the value of the deflection at the tip. The deflection at the tip can be taken from a finite element model or it can be from a static load test. If the deflection at the tip is taken from a finite element model, the errors / uncertainties introduced by the modeling of the joints will still be present. If the deflections are measured during a static load test, it indirectly accounts for these errors. As the intention is to establish the methodology, other sources of errors need to be minimized and hence static load tests were conducted and the deflections measured were used instead of the numerically estimated deflections. The methodology is applied to two typical spacecrafts. The frequencies of the fundamental bending modes of these spacecrafts are estimated from their static deflections with values of α determined using the expressions derived in this work. As a validation of the methodology, these frequencies are compared with the frequencies of the spacecraft obtained through a resonance search test.

2. STRUCTURE WITH CONCENTRATED MASSES

Consider a cantilever of length l. Assume that the cross section is uniform having an area moment of inertia of I. The cantilever is made of material having Young's modulus E. Assume that the mass of the cantilever is negligible. The objective is to derive an expression for determining the parameter α. As mentioned before, the value of α was 1.0 for a cantilever with a tip mass. In a spacecraft, the mass will not be concentrated only at the tip, instead it could be distributed over several locations. The deflections are normally measured at the tip. The objective is to relate the fundamental bending mode frequency with the deflection at the tip. Deriving this relation directly for a case with general mass distribution is a difficult task and hence it is sequentially built up.

2.1. Structure with a Single Mass

Let the cantilever be of length $l + a$ with the mass M being attached at a distance of l from the fixed end. The natural frequency of such a cantilever is:

$$\omega_n{}^2 = \frac{3EI}{Ml^3}. \tag{3}$$

The deflection at the tip is:

$$\delta_{\text{tip}} = \frac{Mgl^3}{3EI} + \frac{Mgl^2}{2EI}a. \tag{4}$$

Writing the natural frequency in the form of Eq. (2), the parameter α can be shown to be:

$$\alpha = 1 + \frac{3a}{2l}. \tag{5}$$

2.2. Structure with Two Masses

Assume that the cantilever has two concentrated masses, with one mass being at the tip. The mass M_1 is at a distance of l_1 from the fixed end and M_2 is at a distance of l_2 from the fixed end. The mass M_2 was at the tip. Using the postulate by Dunkerley,[15,16] the fundamental natural frequency of the beam ω_1 is given by:

$$\frac{1}{\omega_1{}^2} = \frac{1}{\omega_{11}{}^2} + \frac{1}{\omega_{22}{}^2}; \tag{6}$$

where ω_{11} is the natural frequency of the beam when only M_1 is present (that is, all other masses are absent) and ω_{22} was the natural frequency if only M_2 is present (that is, all other masses are absent). Applying this methodology to the present case, the expression for the natural frequency became:

$$\omega_1{}^2 = \frac{3EI}{M_1 l_1{}^3 + M_2 l_2{}^3}. \tag{7}$$

The deflection at the tip due to the weight of the masses attached is:

$$\delta_{\text{tip}} = \frac{M_1 g l_1{}^3}{3EI} + \frac{M_2 g l_2{}^3}{3EI} + \frac{M_1 g l_1{}^2}{2EI}(l_2 - l_1). \tag{8}$$

Relating Eqs. (6) and (7) we get:

$$\alpha = 1 + \frac{3}{2}\frac{M_1 l_1{}^2 (l_2 - l_1)}{M_1 l_1{}^3 + M_2 l_2{}^3}. \tag{9}$$

Eq. (9) reduces to Eq. (5) if we set $M_2 = 0$, $l_1 = l$ and $l_2 = l_1 + a$.

Let us assume that the mass M_2 is not at the tip of the cantilever. The mass M_1 is mounted at a distance of l_1 from the fixed end and M_2 is at a distance of l_2 from the fixed end. Assume that the tip of the cantilever is at a distance of a from M_2. This means that the length of the cantilever is $l_2 + a$. The natural frequency of such a system can be determined using Eq. (7). However, the expression for the deflection at the tip gets modified as:

$$\delta_{\text{tip}} = \frac{M_1 g l_1{}^3}{3EI} + \frac{M_2 g l_2{}^3}{3EI} + \frac{M_1 g l_1{}^2}{2EI}(l_2 - l_1 + a) + \frac{M_2 g l_2{}^2}{2EI} a. \tag{10}$$

Consequently, the parameter α is given by:

$$\alpha = 1 + \frac{3}{2}\frac{M_1 l_1{}^2 (l_2 - l_1 + a) + M_2 l_2{}^2 a}{M_1 l_1{}^3 + M_2 l_2{}^3}. \tag{11}$$

If we set $a = 0$, Eq. (11) reduces to Eq. (9).

If we denote the length of the beam as l, the expression for α can be cast in a convenient form as:

$$\alpha = 1 + \frac{3}{2}\frac{M_1 l_1{}^2 (l - l_1) + M_2 l_2{}^2 (l - l_2)}{M_1 l_1{}^3 + M_2 l_2{}^3}. \tag{12}$$

2.3. Structure with Several Masses

Consider a cantilever having length l. Assume that several masses are attached at different locations on the cantilever. Let the mass M_1 be attached at a distance of l_1 from the fixed end and M_2 be attached at a distance of l_2 from the fixed end etc. The expression for the fundamental natural frequency of such a cantilever is:

$$\omega_1{}^2 = \frac{3EI}{\sum M_i l_i{}^3}. \tag{13}$$

The deflection at the tip of it is given by:

$$\delta_{\text{tip}} = \frac{\sum M_i g l_i{}^3}{3EI} + \frac{\sum M_i g l_i{}^2}{2EI}(l - l_i). \tag{14}$$

Therefore, the parameter α becomes:

$$\alpha = 1 + \frac{3}{2}\frac{\sum M_i l_i{}^2 (l - l_i)}{\sum M_i l_i{}^3}. \tag{15}$$

Equation (15) can be used to determine the parameter α of a cantilever with several masses attached. If the deflection at the tip of the structure for a gravity load of 1 g is known, the frequency of the fundamental bending mode can be estimated using Eq. (2) incorporating the value of α obtained through Eq. (15). This methodology is later demonstrated with two examples.

3. STRUCTURE WITH UNIFORM AND CONCENTRATED MASSES

In many situations, a part of the load will be acting as a uniform load along the length of the beam, in addition to the concentrated masses. The uniform mass is not due to the weight of the cylinder, but due to its continuous connection to the equipment panels.

3.1. Cantilever with Uniform Load

Consider a cantilever with mass per unit length ρ. The natural frequency of this cantilever is[1,2]:

$$\omega_n{}^2 = 3.52^2 \frac{\text{EI}}{\rho l^4}. \tag{16}$$

The deflection at the tip is:

$$\delta_{\text{tip}} = \frac{\rho g l^4}{8EI}. \tag{17}$$

Expressing the natural frequency in terms of deflection at the tip:

$$\alpha = 1.55; \tag{18}$$

which is a well-known[5] result.

Assume the cantilever with a total length of $l + a$ is loaded to a distance of l from the fixed end. The natural frequency of this cantilever is given by Eq. (16). The deflection at the tip is:

$$\delta_{\text{tip}} = \frac{\rho g l^4}{8EI}\left(1 + \frac{8}{6}\frac{a}{l}\right). \tag{19}$$

Expressing the natural frequency in terms of deflection at the tip, for this cantilever:

$$\alpha = 1.55\left(1 + \frac{8}{6}\frac{a}{l}\right). \tag{20}$$

3.2. Cantilever with Uniform Load and Point Mass

Assume that the cantilever is loaded uniformly to a distance of b from the fixed end. A point mass of M is attached at a distance of l from the fixed end. The total length of the cantilever is $l + a$. The natural frequency of this cantilever can be obtained as[15,16]:

$$\frac{1}{\omega_1{}^2} = \frac{Ml^3}{3EI} + \frac{\rho b^4}{3.52^2\text{EI}}. \tag{21}$$

The deflection at the location where the load ends is:

$$\delta_\rho = \frac{\rho g l^4}{8EI} + \frac{\text{Mg}}{3EI}\left(\frac{3lb^2}{2} - \frac{b^3}{2}\right). \tag{22}$$

The deflection at the tip is:

$$\delta_{\text{tip}} = \frac{M\text{g}l^3}{3EI}\left(1 + \frac{3a}{2l}\right) + \frac{\rho g b^4}{8EI}\left(1 + \frac{8}{6}\frac{(l + a - b)}{b}\right). \tag{23}$$

Expressing the natural frequency in terms of deflection at the tip, for this cantilever:

$$\alpha = \frac{\frac{Mgl^3}{3}\left(1+\frac{3a}{2l}\right)+\frac{\rho g b^4}{8}\left(1+\frac{8}{6}\,\frac{(l+a-b)}{b}\right)}{\frac{Ml^3}{3}+\frac{\rho b^4}{3.52^2}}. \quad (24)$$

3.3. Cantilever with Uniform Load and Several Masses

Consider a cantilever with a uniform mass along with two masses attached. Assume that the cantilever is loaded uniformly to a distance of b from the fixed end. A point mass of M_1 is attached at a distance of l_1 from the fixed end and another point mass of M_2 is attached at a distance of l_2 from the fixed end. The total length of the cantilever is $l_2 + a$. The natural frequency of this cantilever is given by:

$$\frac{1}{{\omega_1}^2} = \frac{M_1{l_1}^3}{3EI} + \frac{M_2{l_2}^3}{3EI} + \frac{\rho b^4}{3.52^2\mathrm{EI}}. \quad (25)$$

The deflection at the tip is:

$$\delta_{\text{tip}} = \frac{M_1 g{l_1}^3}{3EI}\left(1+\frac{3\,(l_2-l_1+a)}{2l_1}\right)+\frac{M_2 g{l_2}^3}{3EI} + \frac{M_2 g{l_2}^2}{2EI}a + \frac{\rho g b^4}{8EI}\left(1+\frac{8}{6}\,\frac{(l_2-b+a)}{b}\right). \quad (26)$$

Denoting the total length as l:

$$\delta_{\text{tip}} = \frac{M_1 g{l_1}^3}{3EI}+\frac{M_1 g{l_1}^2}{2EI}(l-l_1)+\frac{M_2 g{l_2}^3}{3EI} + \frac{M_2 g{l_2}^2}{2EI}(l-l_2) + \frac{\rho g b^4}{8EI}\left(1+\frac{8}{6}\,\frac{(l-b)}{b}\right). \quad (27)$$

Expressing the natural frequency in terms of deflection at the tip, following a series of complex algebraic operations:

$$\alpha = 1+\frac{3}{2}\,\frac{M_1{l_1}^2(l-l_1)+M_2{l_2}^2(l-l_2)+\frac{\rho b^3}{3}(l-0.736b)}{M_1{l_1}^3+M_2{l_2}^3+0.242\rho b^4}. \quad (28)$$

Now assume that there are several masses attached to the cantilever. Let the mass M_1 be at distance l_1 from the fixed end and M_2 be at distance l_2 from the fixed end and so on, as shown in Fig. 1. The total length of the cantilever was l. Assume that the cantilever is loaded uniformly to a distance of b from the fixed end. The expression for the natural frequency is:

$$\frac{1}{{\omega_1}^2} = \frac{\sum M_i{l_i}^3}{3EI} + \frac{\rho b^4}{3.52^2\mathrm{EI}}. \quad (29)$$

The parameter α is given by:

$$\alpha = 1+\frac{3}{2}\,\frac{\sum M_i{l_i}^2\,(l-l_i)+\frac{\rho b^3}{3}(l-0.736\,b)}{\sum M_i{l_i}^3+0.242\,\rho b^4}. \quad (30)$$

Equation (30) is the most general expression for determining α, where the cantilever had a uniformly distributed mass along with several concentrated masses attached. In this work, transverse shear effects and the rotary inertia[17–19] were not considered. Incorporation of the above lead to a very complex relation and hence not attempted here. The influence of the above parameters on the first mode was thought to not be that significant as the ratio of the length of the cylinder to the effective cross-sectional area was quite large.

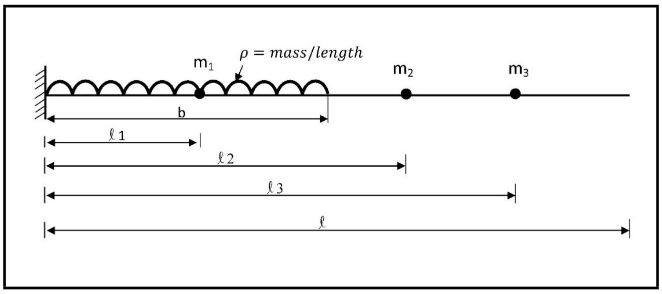

Figure 1. Cantilever with masses attached.

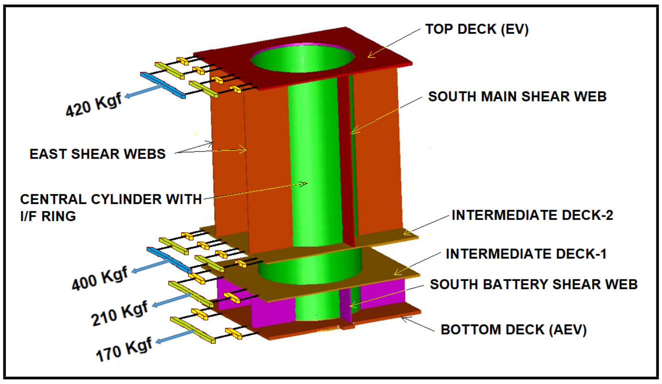

Figure 2. Schematic of the structure of spacecraft 1 and static loading.

4. EXAMPLE: SPACECRAFT 1

A spacecraft having a mass of about 3400 kg was considered as one example, referred here as spacecraft 1. The frequency of the fundamental lateral mode of this spacecraft determined through a resonance search test was 13.7 Hz, details are given later.

4.1. Details of the Spacecraft

Figure 2 shows a schematic of the structure of the spacecraft. It consisted of a central cylinder having diameter of approximately 1200 mm and length 3100 mm. The cylinder was made of honeycomb sandwich construction, the honeycomb core being 12 mm thick. The face sheets were made of CFRP (Carbon Fiber Reinforced Plastics) laminates. Each face sheet had a thickness of approximately 0.9 mm at the bottom and reduces progressively to the top. The first bending mode of the central cylinder was the fundamental lateral mode of vibration of the spacecraft.

There were four horizontal decks connected to the cylinder, as shown in Fig. 2. They were: Bottom Deck (at a distance of 90 mm from the base), Intermediate Deck 1 (at a distance of 547 mm from the base), Intermediate Deck 2 (at a distance of 1001 mm from the base) and Top Deck (at a distance of 3106 mm from the base). The lateral loads acting on the vertical decks were transferred to the cylinder through these panels and they will be acting on the cylinder at the respective interfaces. Thus, there will be 4 concentrated loads acting on the cylinder. They were 170 kg, 210 kg, 400 kg and 420 kg at the interfaces of the bottom deck, intermediate deck 1, intermediate deck 2, and the top deck.

Apart from the above, two propellant tanks were mounted inside the cylinder. Each tank was connected to the cylinder at 24 points that lie in one plane. The lateral load on the tank was transferred to the cylinder as a tangential load. The oxidizer tank, having a mass of 1228 kg, was mounted at a distance

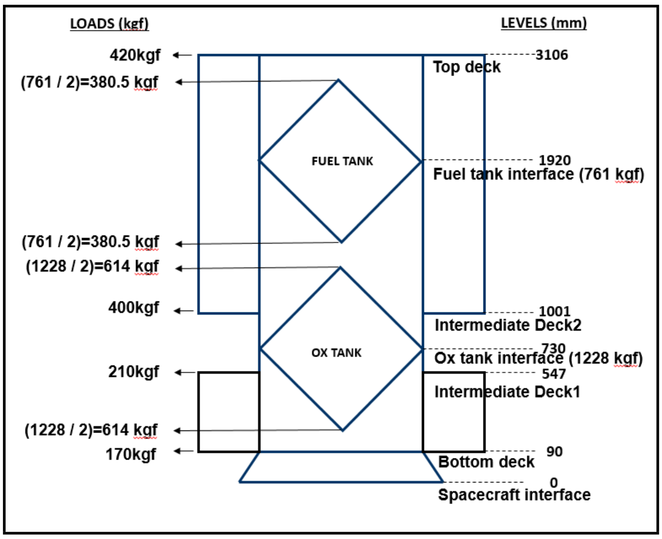

Figure 3. Loads and load application points in spacecraft 1.

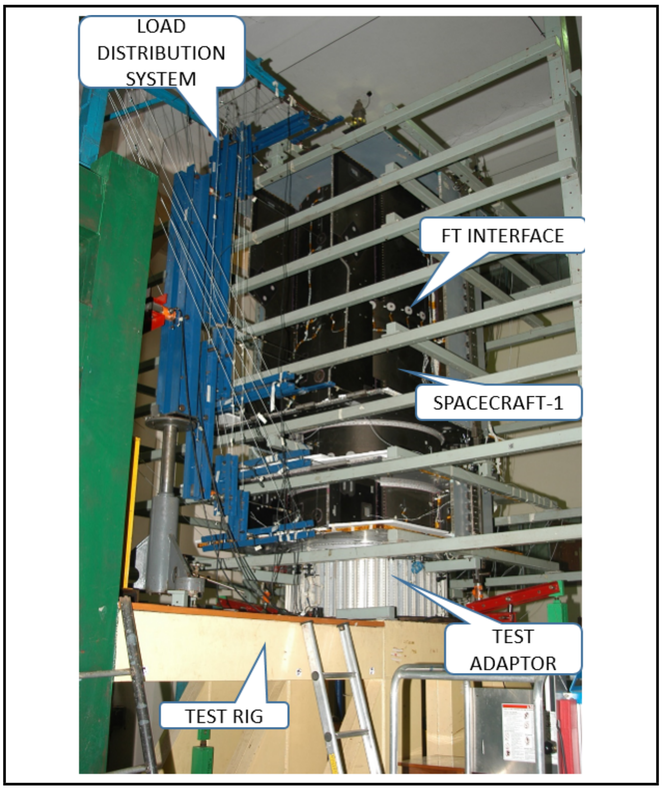

Figure 4. Static load test setup for spacecraft 1.

of 730 mm and the fuel tank, having a mass of 761 kg, was mounted at a distance of 1920 mm from the base of the spacecraft. The cylinder of the spacecraft was subjected to concentrated loads at 6 planes. The masses of some elements such as the cylinder and shear panels amounting to 243 kg can be considered as a UDL. Details of the concentrated masses and their application points are shown in Fig. 3 and Table 1. The total mass of the spacecraft is 3432 kg.

4.2. Details of Static Load Tests

Figure 4 shows the static load test setup. The structure was mounted on a test rig. The satellite is normally assembled to the launch vehicle adaptor using a clamp band. To facilitate the assembly of the clamp band during the static test, a test adaptor was introduced between the spacecraft and the test rig. The loads were applied using hydraulic jacks and distributed to various points through whiffle-tree system. The elements of the test system are indicated in Fig. 4. Similar techniques[20] are employed by other spacecraft industries.

The test structure consisted of the central cylinder with an interface ring, four horizontal decks and the shear panels. It did not include the vertical decks. The direction of loading is shown in Fig. 2. The spacecraft did not have any load-carrying vertical decks placed normal to the direction of loading. There were two vertical decks orthogonal to it called the North and South decks, which carried most of the spacecraft equipment. These vertical decks were connected to the cylinder through shear panels designated as the North and South shear panels.

When a spacecraft undergoes an acceleration along the direction of the loading as shown in Fig. 2, they act normal to the North and South shear panels. As shear panels are very flexible for loads normal to them, the loads from the vertical decks were transferred to the cylinder through the horizontal decks, as in-plane loads in the horizontal decks. Hence during the static load test, the loads of the vertical decks were applied at the horizontal decks as in-plane loads. Thus, the lateral load applied on the horizontal decks is the sum of the load that was acting directly on the horizontal deck and the part of the load of the vertical deck which gets transferred to the horizontal decks. Figure 2 shows the loading on the horizontal decks. Distribution of the load of the vertical deck among the horizontal decks is arrived at based on a finite element model.

The propellant tanks were connected to the cylinder at several locations through a set of radially flexible attachments called tabs (see Fig. 5). The loads that act on tanks were transferred to the cylinder at their respective interfaces. The actual fuel and oxidizer tanks were not used during the static testing, instead tank simulators having interfaces identical to the actual tanks in terms of flexibility were used (Fig. 5). Loads were applied at the top and bottom of the tank simulator (see Figs. 5 and 3). Distribution of force at the interfaces was verified whether they are as in the flight spacecraft, through a finite element model.

The static load tests were conducted on the spacecraft structure simulating launch loads. As the launch loads act along longitudinal and lateral directions simultaneously, the static loading was in the combined fashion. To determine the natural frequency, the deflection under a 1 g equivalent static load with the loads acting only in the lateral direction was needed. There were some characterization tests done which will have only the lateral loads. Results of those tests were selected where only the lateral loads were applied and used for determining the natural frequency. The loads applied during those tests need not correspond to 1 g. In the case of the present spacecraft, a static load of 0.6 g, which corresponds to a load of 1913 kgf was applied. From the results for 0.6 g, the deflection for 1 g is determined and used for determining the natural frequency.

The loads were measured using appropriate load cells. The load cell had an accuracy of ± 0.5 % of the range. For example, the accuracy of the load cell having the range 5000 kgf was ± 25 kgf. It was not possible to apply the uniformly distributed load. Thus, the total load considered corresponded to 3189 kg. Error caused by the absence of this load during the static load test is discussed later.

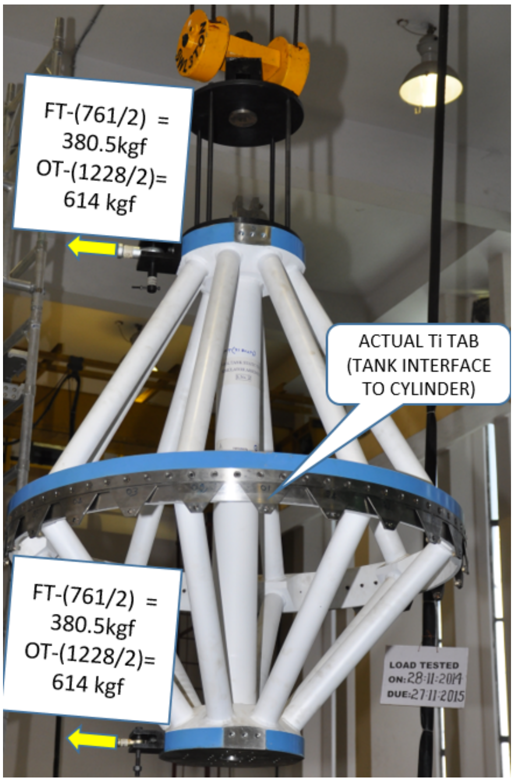

Figure 5. Propellant tank simulator for static test.

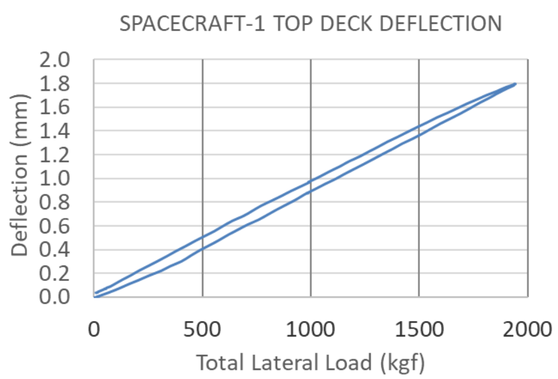

Figure 6. Measured deflection at the tip of spacecraft 1.

4.3. Measured Static Deflection

The deflections were measured using Linear Variable Differential Transducers (LVDT). LVDTs were mounted on a frame which was isolated from the loading frame. These LVDTs had an accuracy of 20 microns and measured up to 10 mm.

A total load of 1913 kgf, which corresponds to an acceleration of 0.6 g was applied. The deflections at four locations on the top deck were measured and the average value of 1.7 mm was taken as the deflection at the tip of the spacecraft. Figure 6 shows the measured deflection at a typical location. Correcting for the deformation due to the test rig and adaptor, which is explained later, the deflection at the tip was 1.3 mm. Correspondingly, the deflection for 1 g was 2.17 mm.

As mentioned before, the uniformly distributed load corresponding to 243 kg was not applied during the static load test due to practical limitations. Deflection due to UDL also needed to be added to the measured deflection. Deflection at the tip of the structure due to this load was theoretically estimated. The deflection estimated using a finite element based mathematical model and it was found to be about 0.1 mm. This deflection was negligible compared to the deflection of 2.17 mm caused by other loads. For determining the natural frequency, the deflection that needed to be considered was that of the cylinder. As load was applied on the deck, the measured deflection included the displacement of the deck due to this in-plane load. This was theoretically estimated using a finite element model and it was 0.02 mm. The displacement at the edge of the top deck was computed and the deformation at the cylinder-top deck interface was computed. The difference between them was 0.02 mm compared to the deflection of 2.17 mm in the top deck and hence it was neglected.

Table 1. Determination of α for Spacecraft 1.

Distance from the I/F of spacecraft	Mass (kg)	$M_i l_i^3$	$M_i l_i^2 (1-l_i)$
90	170	0.12	4.15
547	210	34.4	160.8
730	1228	477.7	1555
1001	400	400	844
1920	761	5386	3327
3106	420	11741	0
UDL	243	1761	640
Total	3432	19800	6531

4.4. Fundamental Mode Frequency

This spacecraft can be considered as a cantilever with 6 masses along with a uniformly distributed load, as given in Table 1. The distance to the free end is 3106 mm.

Using the expression derived in this work, i.e., Eq. (30), the value of α can be calculated as 1.49.

For the UDL, the parameters used were, $b = 3.106$ m and $\rho = 78.2$ kg/m. It is to be noted that the value of α was computed considering all the loads, including the UDL. In Table 1, the value given against UDL in the third column was $0.242\,\rho b^4$ and that in the fourth column was $\frac{\rho b^3}{3}(l - 0.736\,b)$.

The frequency of the fundamental mode was estimated as 13.0 Hz, using Eq. (2) with the value of α as 1.49. As mentioned before, the frequency of the fundamental mode of the spacecraft obtained from a resonance search test was 13.7 Hz. As per Dunkerley's postulate,[15,16] the natural frequency determined will be the lower bound, meaning that the estimated natural frequency will be lower than the actual.

If we had used the conventional method ($\alpha = 1$), the frequency would have been estimated as 10.7 Hz. The expression derived in this work allows a closer estimation of the frequency of the fundamental mode.

The derived relation between the frequency and the static deflection does not incorporate the transverse shear deformation and rotary inertia. Incorporation of the above may improve the estimation of the frequency. It is to be noted that the measured static deflection included the transverse shear deformation, but the expression for α did not incorporate them. These points may explain the difference between the estimated natural frequency and the frequency obtained through the resonance search test.

There may be a question that how such a complex system can be represented by a beam model and still get accurate results. It should be noted that the static deflection used in the estimation of the natural frequency was a measured quantity of the complex system which indirectly includes the stiffness of all joints, material properties etc. If we had used the deflection computed using a finite element model, these kinds of errors would have remained.

Thus, the lateral bending mode frequency of a spacecraft can be estimated from the deflection of the primary structure measured during the static load tests, with a reasonable accuracy using the expression derived here for α. This information can be available well before the assembly of the spacecraft.

4.5. Correction for the Flexibility of the Base

If the test rig and the adaptor are quite rigid, there will not be any deformation developed at the interface of the spacecraft. As they are not infinitely rigid, there will be some amount of rotation as well as linear deformation at the base of the structure. The rotation at the base causes a lateral displacement of the structure which is proportional to the distance from the base which is in addition to the deflection due to the bending. The measured deflection will have the above two components as well. To arrive at the deflection due to the bending, these deflections need to be subtracted from the measured deflection.

The deflection due to the rotation of the base can be determined if the angle of rotation is known. This deflection will be equal to the product of the angle of rotation (in rad) and the distance from the base. The rotation at the base is very small and hence difficult to be measured directly. Therefore, this was determined in this work through a different approach. The rotation that is generated at the top of the adaptor results in longitudinal displacements in opposite directions at the diametrically opposite locations at the top of the adaptor. The longitudinal deformation measured at the top of the adaptor was 0.05 mm. As this location was at distance of 600 mm from the center (radius of the cylinder), the rotation at the adaptor top was 0.09×10^{-3} rad. Though this rotation was very small, it produces a lateral movement of 0.3 mm at the topmost point of the structure, which was at a distance of 3106 mm.

The top of the test adaptor has a lateral movement of 0.1 mm. Though very negligible, this displacement also gets added to the measured deflection.

Combining the two, the top of the structure had an additional deformation of 0.4 mm over the deflection due to the bending. Therefore, the deflection due to bending alone was 1.3 mm (1.7 – 0.4).

4.6. Validation of the Methodology

Validation of the proposed methodology can be done by comparing the estimated natural frequency with the natural frequency obtained through a resonance search test.

The resonance search test was performed using shaker systems. The spacecraft was mounted on the slip table of the shaker systems. As the spacecraft interfaces and shaker interfaces were different, a vibration test fixture was employed to mount the spacecraft on the slip table. The setup for such a test is shown in Fig. 7.

The resonance search test was done by applying a sinusoidal excitation of amplitude 0.15 g at the base of the spacecraft. The frequency was swept from 5 Hz to 100 Hz. The acceleration responses on the top deck of the spacecraft were measured and the transmissibility of acceleration is determined. A typical transmissibility plot at the top deck of a spacecraft is shown in Fig. 8. The frequency at which the transmissibility was the maximum is taken as the resonance frequency. The resonance frequency of this specific spacecraft is found to be 13.7 Hz.

Figure 7. Test setup for the resonance search test.

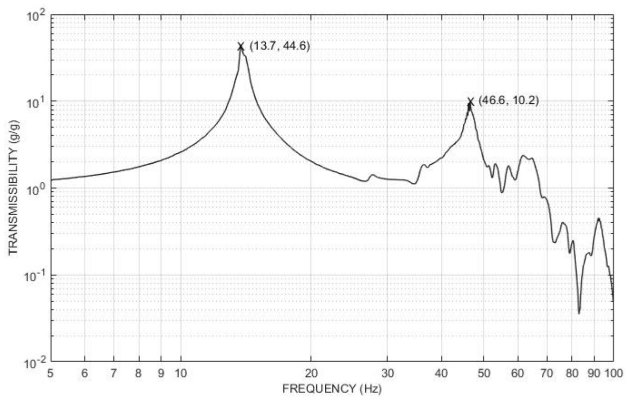

Figure 8. Transmissibility at a typical location in spacecraft.

5. EXAMPLE: SPACECRAFT 2

Another example considered was a spacecraft having a mass of about 5700 kg, referred here as spacecraft 2. The frequency of the fundamental lateral mode of this spacecraft determined through a resonance search test was 11.7 Hz. As the methodology adopted for the resonance search test is identical to that of spacecraft 1, they are not described. The primary bending of the central cylinder was the fundamental lateral mode of the spacecraft in this case as well.

5.1. Details of Spacecraft

The structure of the spacecraft consisted of a central cylinder having diameter of approximately 1200 mm and length 4887 mm. The cylinder was made of honeycomb sandwich construction, the honeycomb core being 12 mm thick. The face sheets were made of CFRP laminates having thickness of approximately 2.5 mm at the bottom and reduces progressively to the top.

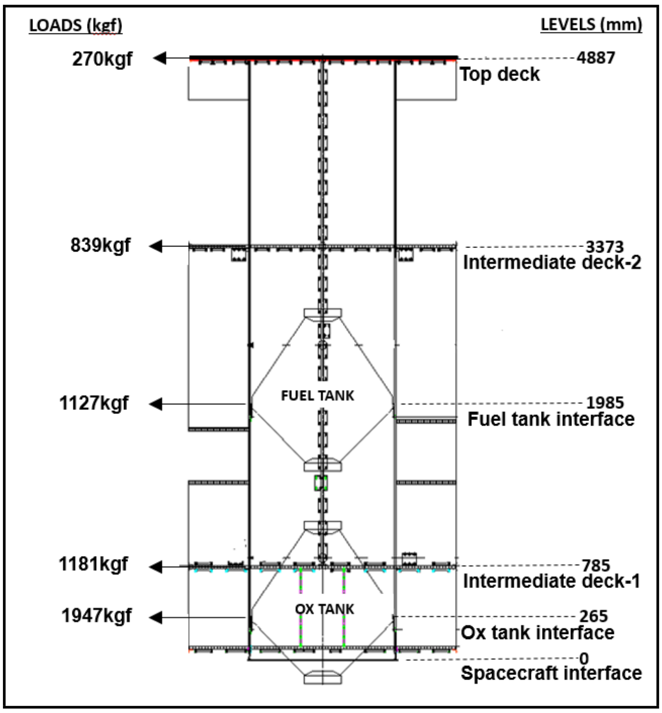

Figure 9. Loads and load application points in spacecraft 2.

There were four horizontal decks (Fig. 9) connected to the cylinder. They were the Bottom Deck (at a distance of 105 mm from the base), Intermediate Deck 1 (at a distance of 785 mm from the base), Intermediate Deck 2 (at a distance of 3373 mm from the base) and the Top Deck (at a distance of 4887 mm from the base). The loads acting on the vertical decks were transferred to the cylinder through these panels.

As in spacecraft 1, two propellant tanks are mounted inside the cylinder. The oxidizer tank had a mass of 1947 kg and was mounted at a distance of 265 mm. The fuel tank had a mass of 1127 kg and was mounted at a distance of 1985 mm from the base of the spacecraft. Thus, the cylinder of the spacecraft was subjected to concentrated loads at 6 locations. As in spacecraft 1, a mass of 293 kg was considered as uniformly distributed load (UDL). Details of the masses and their application points are shown in Fig. 9.

5.2. Details of Static Load Tests

The methodology used for the testing was the same as described for Spacecraft 1. The structure was mounted on a test rig through an adaptor and the loads were applied using hydraulic jacks.

As discussed previously, flexibility of the test rig and the adaptor caused additional deflections. Though these can be distinguished and it was possible to separate them out as demonstrated for Spacecraft 1, the correction should not be quite high. Therefore, the tests rig as well as the adaptor were stiffened for the testing of Spacecraft 2.

The deflections were measured using LVDTs. The deflections were measured at 3 locations on the top deck. These locations are shown in Fig. 10.

It was not possible to apply the uniformly distributed load of 293 kgf. Thus, the total load considered corresponds to 5364 kgf. Due to the absence of this load during the static load test, the error caused was expected to be much less as explained for spacecraft 1. As discussed previously for spacecraft 1, results of the static load tests where only lateral loads

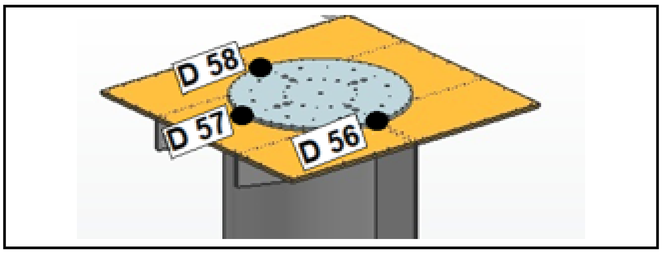

Figure 10. Deflection measurement locations on top deck of spacecraft 2.

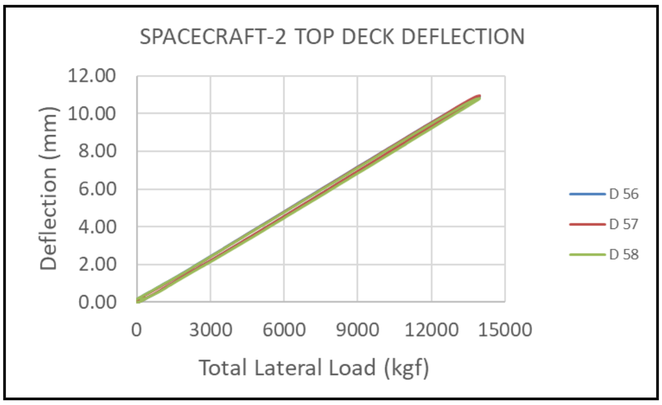

Figure 11. Measured deflection at the top deck of spacecraft 2.

are applied were considered. In this case, it corresponds to 2.6 g with a mass of 5364 kg.

5.3. Measured Static Deflection

The measured deflection at the tip of the spacecraft is 10.9 mm as shown in Fig. 11.

The load-deflection relation was linear. This was true with spacecraft 1 also, though results are not presented. After incorporating the corrections as done in the case of spacecraft 1, the deflection due to bending was 8.9 mm. The deflection for a load that corresponds to a gravity acceleration of 1 g was 3.42 mm.

5.4. Fundamental Mode Frequency

This spacecraft can be considered as a cantilever with 6 masses along with a uniformly distributed load, as given in Table 2. The distance to the free end was 4887 mm.

Applying the expression derived in this work, i.e., Eq. (30), the value of α can be calculated as 1.622.

For the UDL, the parameters used were, $b = 4.887$ m and $\rho = 60$ kg/m. It is to be noted that the value of α was computed considering all the loads, including the UDL. The value given against UDL in the third column of Table 2 is $0.242\ \rho b^4$ and that in the fourth column was $\frac{\rho b^3}{3}(l - 0.736\ b)$.

Table 2. Determination of α for spacecraft 2.

Distance from the I/F of spacecraft	Mass (kg)	$M_i l_i^3$	$M_i l_i^2\ (1{-}l_i)$
265	1947	36	631
785	1181	571	2985
1985	1127	8815	12887
3373	839	32197	14452
4887	270	31513	0
UDL	293	8282	2334
Total	5657	73132 (81414)	30955 (33289)

The frequency of the fundamental mode was estimated to be 10.83 Hz, using Eq. (2) with the value of α as 1.613. As mentioned before, the frequency of the fundamental mode of the spacecraft obtained from a resonance search test was 11.7 Hz. If we had used the conventional method ($\alpha = 1$), the frequency would have been estimated as 8.5 Hz. Thus, well before the spacecraft is realized and the resonance search test is carried out, based on the deflection measured at the tip and incorporating the value of the parameter α determined using the expression derived in this work, the lateral bending mode frequency can be estimated with reasonable accuracy.

6. SUMMARY & CONCLUSIONS

A methodology to determine the fundamental lateral bending mode frequency of a spacecraft with a central cylinder is developed which is based on its lateral deflection. To accomplish this, an expression relating the fundamental lateral mode frequency and the deflection at its tip is derived considering a spacecraft with masses attached at several locations. The most preferred deflection to be considered is that at the tip of the spacecraft, measured during a static load test. The measured deflection during a static load test will have additional components due to the flexibility of the test rig and the test adaptor. These components can be differentiated through suitable measurements at relevant locations and the deflection due to bending alone can be determined. This methodology is demonstrated for a 3000 kg class and a 6000 kg class spacecraft. The frequencies that are estimated from the deflections are compared with the frequencies obtained through a resonance search test on each spacecraft. It is seen that the fundamental bending mode frequency of the spacecraft can be conveniently and accurately estimated based on the deflection measured during static load tests on the primary structure. The frequencies estimated through this methodology are found to be about 0.8 Hz lower than the frequencies of the spacecraft obtained through the resonance search tests.

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