Original Derivation of the Modal Decomposition Analysis for Solving Mixed Boundary-Initial Value Problems

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A semi-analytical formulation is proposed to take into account the transient aspect before a stationary regime is established in a bounded linear inhomogeneous elastic structure. This paper using an original approach introduces the well-known modal decomposition analysis for solving mixed boundary-initial value problems.

1. INTRODUCTION

The finite element method is one of the most powerful methods of numerical analysis that exists today for solving partial differential equations. The method has also become a standard tool in the industrial area. Despite its success in solving elliptic, parabolic, and first order hyperbolic problems, it remains inefficient for second order hyperbolic problems typically encountered in wave propagation.

An important class of problems encountered in mechanics is connected to bounded domains and their modes of vibration. These particular problems have been studied by means of the so-called modal decomposition analysis[1] by mechanical engineers since the fifties[2].

Although it should be noted that a complete description of three-dimensional inhomogeneous bodies using such a method as the solution of boundary-initial value problem is not new,[3] the formulation derived in this paper seems to be efficient for numerical simulations.

Consider elastic wave propagation in the framework of elastodynamics.[4] Most mathematical proofs of the mentioned results can be found, for instance, in applied mathematical textbooks related to finding the numerical solution of partial differential equations by using the finite element method.[5,6] Applications in mechanics, and in physics in general, are considerable and give rise to the publication of new manuscripts every year.[7,8]

Subsequent developments in this paper are concise and brief. Given the solutions associated with the stationary and quasistatic state, the transient aspect is taken into account analytically by solving the wave equation using an eigenfunction expansion.

It is hoped this formulation will prove useful for mechanical engineers confronted with vibration problems, especially as mixed boundary conditions are imposed.

2. FORMULATION

Let us consider a region R which can be occupied, for instance, by either a heterogeneous material like a composite material on a small scale, or complex structures like aircraft on a larger scale, bounded by a surface S. Assume on elastodynamic state $S=[\mathbf{u},\boldsymbol{\sigma}]\in E(\mathbf{f},\rho,\mathbf{c},R\times T^{+})$, where E represents the class of all elastodynamic states. The vector-valued function $\mathbf{u}$ is the displacement field, the symmetric second-order-tensor-valued function $\boldsymbol{\sigma}$ is the stress field, the vector $\mathbf{f}$ is the body force, the scalar ρ is the mass density, and the symmetric fourth-order tensor $\mathbf{c}$ is the stiffness tensor with twenty-one independent components c_{ijkl}. All these quantities are functions of the position of point $\mathbf{x}(x_1,x_2,x_3)$ of the region R in an orthogonal coordinate system. They are also time-dependent, except for the material parameters ρ and c_{ijkl}. The domain T^{+} represents the time interval $[0,+\infty]$.

The state S is the solution of the partial differential equation:

$$\begin{cases} \nabla\cdot\boldsymbol{\sigma}+\mathbf{f}=\rho\dfrac{\partial^2\mathbf{u}}{\partial t^2}; \\ \sigma_{ij}=c_{ijkl}u_{kl}=\dfrac{1}{2}c_{ijkl}\left(\dfrac{\partial u_k}{\partial x_l}+\dfrac{\partial u_l}{\partial x_k}\right), \end{cases} \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} &\mathbf{u}(\mathbf{x},t)=\mathbf{U}(\mathbf{x},t) && \text{on } \Sigma_1\times T^{+}, \\ &\boldsymbol{\sigma}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)=\mathbf{T}(\mathbf{x},t) && \text{on } \Sigma_2\times T^{+}, \Sigma_1\cup\Sigma_2=\Sigma, \end{aligned} \tag{2}$$

and the initial conditions

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\ \frac{\partial\,\mathbf{u}(\mathbf{x},0)}{\partial t}=\mathbf{v}_0(\mathbf{x}) \text{ in } R, \tag{3}$$

where the outward unit normal vector to the surface Σ is denoted as $\mathbf{n}$, $\mathbf{U}$ and $\mathbf{T}$ respectively are the imposed displacement and traction vectors, t is the time, and ∇ is the divergence operator. Whenever appropriate, either bold face symbols or subscript notation verifying Einstein's convention are used to indicate vector and tensor-valued quantities. Equations (1) define a mixed boundary-initial value problem.

3. SOLUTION

A standard approach for such problems consists of splitting the original elastodynamic state $S=S^{(1)}+S^{(2)}$ into $S^{(1)}=[\mathbf{u}^{(1)},\boldsymbol{\sigma}^{(1)}]$ and $S^{(2)}=[\mathbf{u}^{(2)},\boldsymbol{\sigma}^{(2)}]$, where $S^{(1)}$ is the solution of the partial differential equation with homogeneous